Summary

A classical Skolem sequence is a sequence $a_1, a_2, \ldots, a_{2n}$ where $a_i \in A = \{1 \ldots n\}$. Each a_i occurs exactly twice in the sequence and the two occurrences are exactly a_i positions apart. A set A that can be used to produce classical Skolem sequences is called a classical Skolem set. The problem of deciding which sets of the form $A = \{1 \ldots n\}$ that are classical Skolem sets was solved by Thoralf Skolem in the late 1950's. We study the natural generalization where A is allowed to be any set (or even multiset) of n positive integers. We give necessary conditions for the existence of Skolem sets of this generalized form. We conjecture these necessary conditions to be sufficient, and give computational evidence in favor of our conjecture. We prove that the existence question for an even more generalized form of Skolem sets are $\mathcal{NP}$-complete. We answer an existence question for permutations similar to the existence question for classical Skolem sequences. We also look at enumerative aspects, the number of multi Skolem sequences (multisets allowed) of order n are proved to be $(2n-1)!!$.

Contents

1 Introduction

Classical Skolem sequences was introduced by Thoralf Skolem in 1957 [18], for the construction of Steiner triple systems. He looked at sets of the form $A = \{1, 2, \ldots, n\}$ and asked whether you always could form a sequence with two copies of every element k in the set so that the two copies of k was placed k places apart in the sequence. For example the set $\{1, 2, 3, 4\}$ can be used to form the sequence 42324311, but the set $\{1, 2, 3\}$ can not be used to form such a sequence. A close relative to the classical Skolem sequences is the Langford sequences. Langford sequences is equivalent to classical Skolem sequences with the difference that the two copies of k must be placed $k + 1$ places apart. For example the set $\{1, 2, 3\}$ can be used to construct the sequence 312132. Dudley Langford published an article in 1958 [8] where he managed to construct such sequences from the sets $\{1 \ldots n\}$ with $n = 5, 6, 9$ and 10.

Note that today people usually define a **Langford sequence** to be a classical Skolem sequence which starts late. That is a sequence that is constructed from a set of the form $\{a, a + 1, \ldots, b - 1, b\}$. The example of a Langford sequence from above becomes in this notation 423243.

Since these early papers appeared many different aspects of classical Skolem sequences and Langford sequences have been studied. One reason for them being so well studied is that they have important applications in several branches of mathematics, most notably in design theory and graph labeling, see [15]. One real world application of the generalization of classical Skolem sequences we are about to study is the generation of binary sequences with controllable complexity [6].

A natural generalization of classical Skolem sequences that is not very well studied is when we allow the set of numbers used to generate the sequences to be any set or even multiset of positive integers. We call such sequences **Skolem sequences**. This generalization is the main topic of this thesis.

We call a set that can be used to construct a Skolem sequence a **Skolem set**. For example the set $\{2, 3, 5, 6\}$ is a Skolem set since it can be used to generate the sequence 56232536. When we want to point out that the sequences are constructed from multisets we call the sequences **multi Skolem sequences** and the corresponding multiset for **multi Skolem set**. We define the order of a Skolem sequence to be the cardinality of the corresponding Skolem set.

We now state the most important known results that are related to the generalization that we are about to study.

Theorem 1.1 ([18]) *$\{1, 2, 3, 4, \ldots, n-1, n\}$ is a Skolem set if and only if $n \equiv 0$ or 1 (mod 4).*

The preceding theorem classify which sets that can be used to construct classical Skolem sequences.

Theorem 1.2 ([17]) *A set $A = \{a, a + 1, a + 2, \ldots, b - 1, b\}$ is a Skolem set if and only if*

1. $|A| \geq 2a - 1$ *and*

2. $|A| \equiv 0$ *or* 1 $(mod\ 4)$ *when* a *is odd,* $|A| \equiv 0$ *or* 3 $(mod\ 4)$ *when* a *is even.*

The preceding theorem classify which sets that can be used to construct Langford sequences.

Theorem 1.3 ([14]) *A set* $\{1, 2, \ldots, m-1, m+1, \ldots, n\}$ *is a Skolem set if and only if* $n \equiv 0, 1$ *(mod 4) and* m *is odd, or* $n \equiv 2, 3$ *(mod 4) and* m *is even.*

Sequences constructed from these sets in the preceding theorem are known in the literature as near-Skolem sequences, they have been used to construct cyclic Steiner triple systems.

2 Necessary conditions

We give two necessary conditions for a set A to be a Skolem set. These two necessary conditions are then conjectured to be sufficient for a set A to be a Skolem set.

Theorem 2.1 *A necessary condition for a set* A *of positive integers to be a Skolem set is that the number of even elements in* A *is even.*

Proof: Define the function ϕ to be the sum of the number of elements between the pairs of numbers in the sequence. For example ϕ of 134341 is $4 + 1 + 1 = 6$, because the number of elements between the pair of ones is 4, the number of elements between the pair of threes is 1 and the number of elements between the pair of fours is 1. If we let two adjacent elements in the sequence be transposed, one of the following four cases occurs:

1. Both elements are taken one step closer to their partner, ϕ is therefore decreased by 2.

2. Both elements are taken one step further away from their partner, ϕ is therefore increased by 2.

3. One of the elements is taken one step closer to its partner, while the other element is taken one step further away from its partner, ϕ remains unchanged.

4. The adjacent elements are partners, ϕ remains unchanged.

We see that the parity of ϕ remains unchanged when we transpose adjacent elements in the sequence. It is also easy to see that given a sequence S with elements taken from A we can construct all possible sequences with elements from A by repeated transposition of adjacent elements. Clearly we can construct the sequence where every element in A is adjacent to its partner, ϕ is then 0 (even parity). Thus it follows that a Skolem sequence must have a ϕ that is even. In a Skolem sequence a pair of odd numbers contribute an even number to ϕ but a pair of even numbers contribute an odd number to ϕ. Since ϕ always is even it follows that the number of even numbers in a Skolem set A must be even.

□

Note that the proof is easily seen to hold also when we allow A to be a multiset.

Theorem 2.2 *Another necessary condition for a set A to be a Skolem set is that it is dense enough. Let $n = |A|$, also assume that the elements in A are sorted in decreasing order. That is, $A = (a_1, a_2 \dots a_n)$ where $a_i > a_j$ when $i < j$. A is dense enough if and only if the following inequality is satisfied for all $m \leq n$:*

$$\sum_{i=1}^{m} a_i \leq m(2n - m)$$

Proof: Given a Skolem sequence, we think of the numbers between a pair as fillers to that pair. We put the number of times a number appears as a filler in the exponent to that number. For example $7^0 3^1 4^2 2^3 3^3 2^2 4^1 7^0$, the first occurrence of 3 is only filler to the pair of sevens, therefore it is given the exponent 1 and so on. The maximum sequence of exponents in a sequence generated from a set A with $|A| = n$ is: $0, 1, 2, \dots, n-1, n-1, n-2, \dots, 2, 1, 0$. Note that by maximum sequence we mean that it has the largest possible value at all positions. Now look at the m largest elements in A, they can together take at most $\sum_{i=1}^{m} 2(n-i)$ elements as fillers (according to the maximum sequence of exponents). The sum $\sum_{i=1}^{m}(a_i - 1)$ can be interpreted as the number of fillers required by the m largest elements in A. Clearly a set can not be a Skolem set if the m largest elements requires more fillers then what is maximum available. Thus we have the following inequality:

$$\sum_{i=1}^{m}(a_i - 1) \leq \sum_{i=1}^{m} 2(n - i)$$

Which is equivalent to:

$$\sum_{i=1}^{m} a_i \leq \sum_{i=1}^{m}(2(n - i) + 1)$$

If we write the sum on the right in closed form we get:

$$\sum_{i=1}^{m} a_i \leq m(2n - m)$$

□

Note that this necessary condition also holds when we allow A to be a multiset.

Conjecture 2.3 *We conjecture that the two necessary conditions in Theorems 2.1 and 2.2 are sufficient for a set A to be a Skolem set.*

The conjecture has been verified by computer search for all sets of size 13 or less. The conjecture does not hold when we allow the sets to be multisets. A simple counterexample is the multiset $\{3, 3, 3, 3\}$, it passes both necessary conditions, but can not be used to construct a multi Skolem sequence.

It is interesting to see how the necessary and sufficient conditions in Theorem 1.2 and 1.3 are related to our necessary conditions (Theorem 2.1 and 2.2). We begin with looking at condition 1 in Theorem 1.2 which is a density condition for sets of the form $A = \{a, a+1, a+2, \ldots, b-1, b\}$. We examine what happens when we apply our density condition from Theorem 2.2 to sets of this form. First note that in this case we only need to look at the inequality in Theorem 2.2 with $m = n$, because if the inequality holds when $m = n$, it will hold also when $m < n$ and if it does not hold for some $m < n$ it will not hold for $m = n$ either. Remember that in Theorem 2.2 we defined n to be $|A|$, so in this case $n = b - a + 1$. Thus by Theorem 2.2 we get:

$$\sum_{i=1}^{n} a_i \leq \sum_{i=1}^{n} (2(n-i)+1)$$

If we write the sums in closed form and substitute n with $b - a + 1$, we get:

$$\frac{(a+b)(b-a+1)}{2} \leq (b-a+1)^2$$
$$a + b \leq 2b - 2a + 2$$
$$2a - 1 \leq b - a + 1$$
$$2a - 1 \leq |A|$$

This is exactly condition 1 in Theorem 1.2. Condition 2 in Theorem 1.2 is a parity condition that states that the number of even elements in A must be even, this is exactly our condition from Theorem 2.1. It should come as no surprise that the condition in Theorem 1.3 which is concerned with sets of the form $\{1, 2, \ldots, m-1, m+1, \ldots, n\}$, states that the number of even elements must be even. Note that in this case no density condition is needed because all sets of this form are dense enough. Thus we can conclude that the conditions in Theorem 1.2 and 1.3 are special cases of our conditions in Theorem 2.1 and 2.2 and also that our conditions are sufficient when we have sets of the form $\{a, a+1, a+2, \ldots, b-1, b\}$ or $\{1, 2, \ldots, m-1, m+1, \ldots, n\}$.

It is easy to formulate special cases of Conjecture 2.3, for example for sets A of the form $A \subset \{1, 2, \ldots, n\}$ where $|A| = n - 2$. That is A is a subset of $\{1, 2, \ldots, n\}$ such that exactly two elements $a_i, a_j \in \{1, 2, \ldots, n\}$ are missing from A.

Conjecture 2.4 *We conjecture that a subset A of $\{1, 2, \ldots, n\}$ such that exactly two elements $a_i, a_j \in \{1, 2, \ldots, n\}$ are missing from A, is a Skolem set if and only if the following conditions holds.*

1. *If* $n \equiv 0$ *or* 1 $(mod\ n)$ *then* $a_i \equiv a_j$ $(mod\ 2)$.
2. *If* $n \equiv 2$ *or* 3 $(mod\ n)$ *then* $a_i \equiv a_j - 1$ $(mod\ 2)$.
3. $A \notin \{\{3\}, \{2,4\}, \{2,4,5\}, \{3,4,5,6\}\}$.

Even a proof of this special case of Conjecture 2.3 would be a very nice and publishable result.

3 Enumerative aspects

The problem of deciding how many Langford sequences of a given order there exists goes under the name Langford's problem. It is quite well studied, but so far no closed formula is known and the fastest known algorithm runs in exponential time. For an overview of the results see [11]. Recently Langford's problem has began to be used as a benchmark by the computer science community to evaluate various Constraint Satisfaction techniques [7]. The problem of deciding how many classical Skolem sequences of a given order there exists has not been given the same attention. A table of the known values is available in [15]. We reproduce it here below in Table 1 as a reference.

Order	1	4	5	8	9	12	13
#Sequences	1	6	10	504	2656	455936	3040560

Table 1: Number of classical Skolem sequences.

As for Langford's problem, no closed formula is known and the fastest known algorithm runs in exponential time. We begin our investigation by giving a closed formula for the number of multi Skolem sequences of any given order. Note that we say that a sequence is of order n if the set that it is generated from has cardinality n.

Theorem 3.1 *The number of multi Skolem sequences of order* n *is* $(2n-1)!!$.

Proof: Think of the sequence as a tape with $2n$ empty cells. Pick 2 cells, say i and j, from the $2n$ available cells. Write the number $|i-j|$ in cells i and j. Now repeat the procedure and choose 2 cells from the remaining $2n-2$ cells. Continue in this way until no more empty cells remains. It is clear that the sequence of numbers on the tape is a multi Skolem sequence and that every multi Skolem sequence can be constructed in this way. Note that we are not interested in the order in which the n pairs are chosen, we do not care for example if 2 and 6 were chosen before or after 1 and 3. We compensate for this by putting $n!$ in the denominator. Thus the number of multi Skolem sequences of order n is

$$\frac{\binom{2n}{2}\binom{2n-2}{2}\cdots\binom{4}{2}\binom{2}{2}}{n!} = \frac{2n!}{n!2^n} = (2n-1)(2n-3)\ldots(3)(1) = (2n-1)!!$$

□

Despite our success in finding a closed formula for the number of multi Skolem sequences, we have not been as fortunate when it comes to finding a formula for the number of Skolem sequences (those constructed from ordinary sets). We have put considerable time and effort (approximately 4 months of computation on a 350 MHz Pentium II computer) in computing the number of Skolem sequences of order at most 13. The actual numbers are available in Table 2 below.

Order	#Sequences
1	1
2	1
3	5
4	29
5	145
6	957
7	8397
8	85169
9	944221
10	11639417
11	160699437
12	2430145085
13	39776366397

Table 2: Number of Skolem sequences.

As we will see later this table also gives the number of perfect graceful matchings in K_{2n}, and the number of fixed point free involutions in which the distances of the transposed elements are distinct. As you may have noticed, the number of Skolem sequences of a given order seems to be odd.

Theorem 3.2 *The number of Skolem sequences of a given order is always odd.*

Proof: We can read a Skolem sequence both from right to left and from left to right. Except for the one symmetric sequence (of every order) of the form $\ldots 75311357\ldots$, a Skolem sequence and its reversal will be distinct. We can pair every sequence with its reversal, except for the one that is symmetric. So for any given order we will have $2k+1$ Skolem sequences for some integer k.

□

Another interesting enumerative question is that of the number of Skolem sets of a given order. If a proof of Conjecture 2.3 were found it might be possible to find a closed formula for the number of Skolem sets of any given order. Table 3 contains the number of Skolem sets of orders 1 to 13.

Order	#Skolem sets
1	1
2	1
3	3
4	11
5	35
6	114
7	407
8	1486
9	5414
10	19923
11	74230
12	278462
13	1049318

Table 3: Number of Skolem sets.

4 Related objects

4.1 Perfect matchings

A matching on a graph G is a set of edges in G such that no two of them have a vertex in common. The largest possible matching consists of $\frac{n}{2}$ edges where n is the number of vertices in G, and such a matching is called a perfect matching. Note that not all graphs have perfect matchings. The set of perfect matchings in the complete labeled graph on $2n$ vertices, K_{2n} can be put in one to one correspondence with the set of multi Skolem sequences of order n. Every perfect matching M in K_{2n} can be represented as a multi Skolem sequence in the following way. For every pair (i, j) of matched vertices in M we put $|i - j|$ in position i and j in the sequence, this clearly gives rise to a multi Skolem sequence. A multi Skolem sequence can be represented as a perfect matching in the following way. Start with the element a_1 in position 1 and match vertex 1 with vertex $a_i + 1$ in the graph, remove the elements in the sequence at positions 1 and $a_i + 1$. Continue with the next available element a_j at position j in the sequence and match j with $a_j + j$ in the graph, remove the elements in the sequence at positions j and $a_j + j$. Continue in this fashion until no more elements in the sequence are available. This clearly gives rise to a perfect matching M in the graph. Thus we have established the one to one correspondence. Moreover this together with Theorem 3.1 gives us the well known result that the number of perfect matchings in K_{2n} is $(2n-1)!!$. For an example of a multi Skolem sequence and its corresponding perfect matching see Figure 1 below.

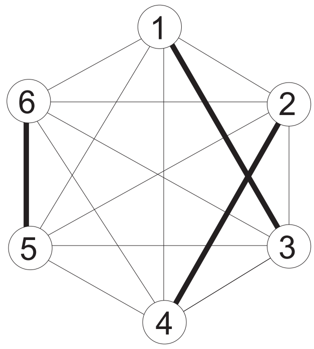

Figure 1: A multi Skolem sequence 222211 and its corresponding perfect matching $(1,3)(2,4)(5,6)$.

4.2 Perfect graceful matchings

Given a graph $G = (V, E)$ where every edge has been given a label, and a perfect matching M on G. If all the edges in M have different labels we say that the perfect matching is graceful. If we label all edges in a labeled graph with the absolute value of the difference between the vertices it connects, then a perfect graceful matching in this graph can be seen as a Skolem sequence. See Figure 2 below for an example. It should be clear that a set A of cardinality n is a Skolem set if and only if K_{2n} with the edges labeled in the way described above and all edges with labels not in A removed, has a perfect graceful matching. We denote graphs constructed in this way G_A. For example the set $A = \{1, 2, 4\}$ is a Skolem set if and only if the graph G_A in Figure 2 has a perfect graceful matching.

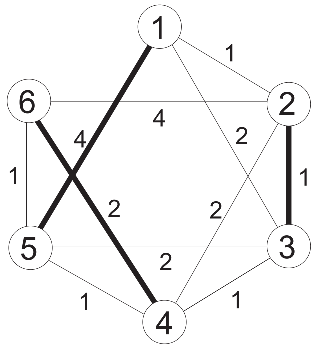

Figure 2: A perfect graceful matching $(1,5)(2,3)(4,6)$ and the corresponding Skolem sequence 411242.

Following the argument above establishing the one to one correspondence between perfect matchings in K_{2n} and multi Skolem sequences of order n gives us a one to one correspondence between Skolem sequences of order n and perfect graceful matchings in K_{2n}. Thus Table 2 gives us the number of perfect graceful matchings in K_{2n} when $n \leq 13$.

It is easy to see that a necessary condition for a graph to have perfect graceful matching is that the number of different edge labels are at least $n/2$, where n is the number of vertices in the graph. This condition is trivially satisfied in graphs G_A constructed from Skolem sets. Another necessary condition for a graph to have a perfect graceful matching is that it has a perfect matching. We will prove that graphs G_A constructed from sets satisfying the necessary conditions in Theorem 2.1 and 2.2 have perfect matchings by investigating a problem about multi Skolem sequences.

Theorem 4.1 *Given a set A of cardinality n such that for all $a_i \in A$, $a_i < 2n$, then there always exists two numbers a_i and a_j in A such that the multiset $B = \{a_i^k, a_j^{n-k}\}$ is a multi Skolem set. The notation a_i^k means k copies of the number a_i, of course k must be $\leq n$.*

Proof: First note that if the number $n \in A$, $B = \{n^n\}$ is a multi Skolem set. For example take $n = 4$, $B = \{4^4\}$, the corresponding multi Skolem sequence is 44444444. So in the following let us assume that $n \notin A$. Color the numbers $\{1, 2, \ldots, n-1, n+1, \ldots, 2n-2, 2n-1\}$ such that for all $k \in \{1 \ldots n-1\}$ the numbers $2n-k$ and $n-k$ get color k. Thus the numbers have been colored in $n-1$ different colors. Now let A be a set of n of these numbers. It follows from the pigeonhole principle that at least two of the numbers in A are of the same color. That is, a pair of the form $2n-k, n-k$. Thus let $a_i = 2n-k$ and $a_j = n-k$. If $2n-1 \in A$ and $n-1 \in A$, $B = \{(2n-1)^1, (n-1)^{n-1}\}$ is a multi Skolem set. For example take $n = 4$, $B = \{7^1, 3^3\}$, the corresponding multi Skolem sequence is 73333337. It should be clear that if $2n-k \in A$ and $n-k \in A$, we can form the multi Skolem set $B = \{(2n-k)^k, (n-k)^{n-k}\}$ with the corresponding multi Skolem sequence $\underbrace{(2n-k)(2n-k)\ldots}_{k\ copies\ of\ 2n-k}\ \underbrace{(n-k)(n-k)\ldots}_{2(n-k)\ copies\ of\ n-k}\ \underbrace{(2n-k)(2n-k)\ldots}_{k\ copies\ of\ 2n-k}$

□

Corollary 4.2 *Every graph G_A constructed from a set A that satisfies both the necessary conditions in Theorem 2.1 and 2.2 has a perfect matching.*

Proof: Theorem 4.1 tells us that G_A has a perfect matching using only edges labeled a_i and a_j.

□

4.3 Restricted involutions

An involution is a permutation which is its own inverse. A fixed point free involution is an involution where no element is mapped to itself. Another way to express this is that a fixed point free involution is a permutation such that when written in cycle notation it only has cycles of length 2. It is clear that this gives us a one to one correspondence between involutions in S_{2n} and perfect matchings in K_{2n} (and thus multi Skolem sequences of order n). This together with Theorem 3.1 gives us the result that the number of fixed point free involutions in S_{2n} is $(2n-1)!!$.

If we restrict the fixed point free involutions and require that all the distances between the transposed elements must be distinct, we have a one to one correspondence between these restricted involutions in S_{2n} and perfect graceful matchings in K_{2n} (and thus Skolem sequences of order n). For example the involution $(15)(23)(46)$ satisfies this condition because $|1-5| = 4$, $|2-3| = 1$ and $|4-6| = 2$. Thus Table 2 gives us the number of these restricted involutions in S_{2n} when $n \leq 13$.

The permutation matrix of a fixed point free involution is symmetric and has the northwest to southeast main diagonal empty. The restriction on the involutions has a nice interpretation in terms of permutation matrices, namely that every northwest to southeast diagonal of the matrix can have at most one entry. Two elements on the same northwest to southeast diagonal implies that two of the transpositions in the involution are of the form (ab) and $((a+k)(b+k))$, and of course $|a-b| = |(a+k)-(b+k)|$, violating the condition. See Figure 3 for an example of a permutation matrix representing an involution satisfying the condition.

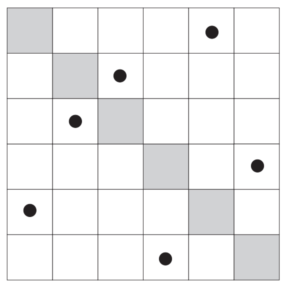

Figure 3: The permutation matrix corresponding to the fixed point free involution $(15)(23)(46)$ (Skolem sequence 411242). The shaded squares indicate forbidden positions.

Given a set A of cardinality n we construct a $2n \times 2n$ matrix M which have certain forbidden positions, namely the main northwest to southeast diagonal and those m_{ij} where $|i-j| \notin A$. Then elements can be added to M to yield a symmetric permutation matrix with at most one element on every northwest to southeast diagonal if and only if A is a Skolem set. See Figure 4 for an example.

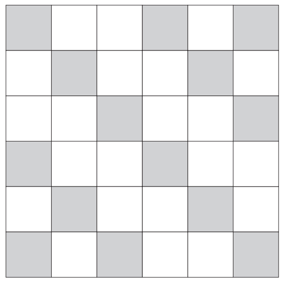

Figure 4: A matrix such that elements can be added to it to yield a symmetric permutation matrix where every northwest to southeast diagonal contains at most one element if and only if the set $A = \{1, 2, 4\}$ is a Skolem set. The shaded squares indicate forbidden positions.

5 NP-completeness of perfect graceful matching

Complexity theory is the theory of classifying problems based on how difficult they are to solve. The two most important classes are $\mathcal{P}$ and $\mathcal{NP}$. $\mathcal{P}$ contains all problems solvable in polynomial time, that is problems that can be solved in $O(n^k)$ steps, where k is a constant and n is the size of the problem. $\mathcal{NP}$ contains all problems for which a provided solution can be verified in polynomial time. One of the most important open problems in theoretical computer science and mathematics is whether $\mathcal{NP}$ equals $\mathcal{P}$ or not. Most people believe that $\mathcal{NP} \neq \mathcal{P}$.

The most difficult problems in $\mathcal{NP}$ are called $\mathcal{NP}$-complete problems. They have the property that if any single one of them is solvable in polynomial time, then all problems in $\mathcal{NP}$ would be solvable in polynomial time, thus proving $\mathcal{NP} = \mathcal{P}$. The easiest way of proving that a problem is $\mathcal{NP}$-complete is that of reducing, in polynomial time, a problem already known to be $\mathcal{NP}$-complete to it. By reducing a problem to another problem we mean giving a constructive transformation that maps any instance of the first problem into an equivalent instance of the second problem. If we have a polynomial time reduction from one problem to another, this ensures that any polynomial time algorithm for the second problem can be converted into a corresponding polynomial time algorithm for the first problem. Thus it is clear that if we can give a polynomial time reduction from a $\mathcal{NP}$-complete problem to another problem, we have proved this other problem to be $\mathcal{NP}$-complete. See [5] for an in-depth treatment of the theory of $\mathcal{NP}$-completeness.

The problem of deciding if a graph has a perfect matching is solvable in polynomial time ($\in \mathcal{P}$). Tutte proved conditions that are both necessary and sufficient for the existence of a perfect matching in a graph. For a proof of Tutte's theorem and a comprehensive treatment of matching theory, see [10]. We would like to find nice necessary and sufficient conditions for the existence of a perfect graceful matching in a graph. Then we should be able to prove that the special type of graphs G_A that has a perfect graceful matching if and only if a certain set A is a Skolem set, satisfies the sufficient conditions. Thus giving us a proof of Conjecture 2.3. Unfortunately such nice necessary and sufficient conditions are unlikely to exist because we show below that the problem of deciding if a graph has a perfect graceful matching is $\mathcal{NP}$-complete. The proof will be a reduction from the well known $\mathcal{NP}$-complete problem 3-Dimensional Matching to Perfect Graceful Matching. See [5] for the details on the $\mathcal{NP}$-completeness of 3-Dimensional Matching.

3-DIMENSIONAL MATCHING (3DM)
INSTANCE: Set $M \subseteq W \times X \times Y$, where W, X and Y are disjoint sets having the same number q of elements.
QUESTION: Does M contain a matching, i.e. a subset $M' \subseteq M$ such that $|M'| = q$ and no two elements of M' agree in any coordinate?

PERFECT GRACEFUL MATCHING (PGM)
INSTANCE: Graph $G = (V, E)$, where every $e \in E$ has been assigned a label.
QUESTION: Does G contain a perfect graceful matching M, i.e. a perfect matching M where all $e \in M$ have different labels?

Theorem 5.1 Perfect Graceful Matching *is $\mathcal{NP}$-complete.*

Proof: First we must prove that Perfect Graceful Matching $\in \mathcal{NP}$, that is that given an instance $G = (V, E)$ of Perfect Graceful Matching and a proposed perfect graceful matching M, we should be able to verify in polynomial time whether M indeed is a perfect graceful matching of G. This is easy, just check that all the labels of $e \in M$ are distinct, $|M| = |V|/2$ and that no vertex is covered by more than one edge. This can all be done in polynomial time.

We can represent an instance of 3-Dimensional Matching as a labeled (multi) graph in the following way: $G = (V, E)$ where $V = W \cup Y$ and for every $(w, x, y) \in M$, add an edge labeled x between w and y. Note that G is a multigraph, i.e. two vertices can be connected by several edges. We must get rid of the multigraph property of G to complete the reduction to Perfect Graceful Matching. But it should be clear that the reduction would be complete now if we allow the instances of Perfect Graceful Matching to be multigraphs. Because the graph G formed in the way described above has a perfect graceful matching if and only if M contains a matching.

One important result on 3-Dimensional Matching that is stated (without proof) in [5] is that 3-Dimensional Matching remains $\mathcal{NP}$-complete even if no element occurs in more than three triples. So we only need to consider (multi) graphs G where all vertices have a maximum degree of 3. Every multigraph G will be reduced to a simple graph G' that has a perfect graceful matching if and only if G has one. We need to handle four different cases, see Figures 5-8 below. In each case we transform a component of the multigraph G (left hand side) into a equivalent component of the simple graph G' (right hand side). All edge labels on the right hand side not occurring on the left hand side are new ones not previously present in G'. The transformation can be performed in polynomial time and in each case it is easy to verify that the multigraph on the left hand side has a perfect graceful matching if and only if the graph on the right hand side has one. This completes our reduction from 3-Dimensional Matching to Perfect Graceful Matching, and thus proves that Perfect Graceful Matching is $\mathcal{NP}$-complete.

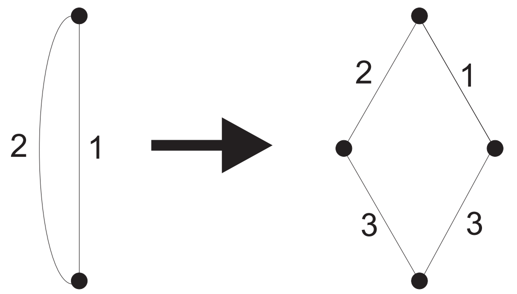

Figure 5: Case 1.

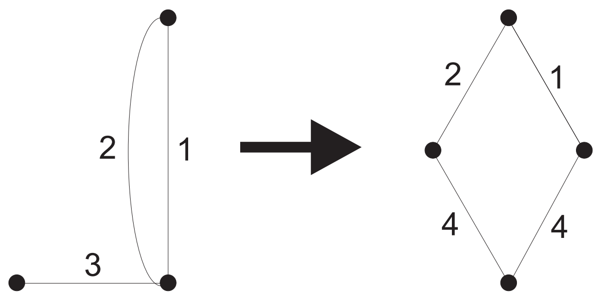

Figure 6: Case 2. (Note that the edge labeled 3 can be removed because it can never be part of a perfect graceful matching.)

$\square$

The problem Perfect Graceful Matching is interesting in its own right. It is easy to come up with real world problems that are instances of Perfect Graceful Matching. Take the standard perfect matching problem of assigning n students to two-person dormitory rooms for example. If we are given a list of pairs of students who would be willing to share rooms, can we assign each person an acceptable roommate? If we represent the students as vertices and add edges between students that find each other acceptable as roommates, it is

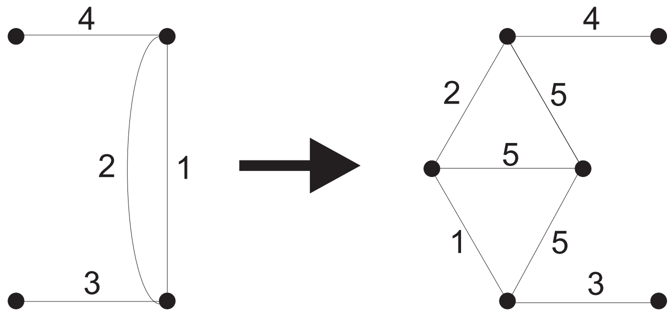

Figure 7: Case 3.

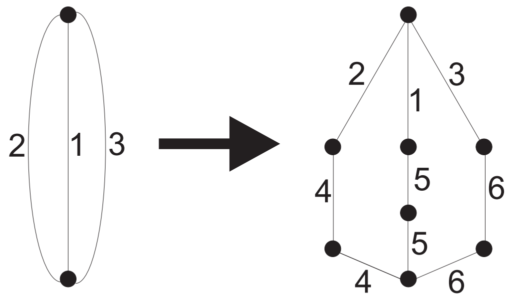

Figure 8: Case 4.

clear that such an acceptable assignment can be done if and only if the graph has a perfect matching. If we allow the list of pairs of students to be supplemented so that each pair of students willing to share rooms specify which dormitory room they would prefer, we are faced with an instance of Perfect Graceful Matching. Form the graph as before, but add the preferred dormitory room as a label to every edge. Then we can assign the students in acceptable pairs such that every pair is given their preferred dormitory room if and only if the graph has a perfect graceful matching.

6 k-extended Skolem sequences

A **k-extended Skolem sequence** is a Skolem sequence which has a hole in position k. We will use the null symbol 0 to represent the hole. If a set A can be used to construct a k-extended Skolem sequence, we will say that $k - A$ is a Skolem set. For example, $3 - \{2, 4\}$ is a Skolem set, because it can be used to construct the 3-extended Skolem sequence 42024. When A is of the form $\{1, 2, \dots, n-1, n\}$ we will call the set $k - A$ a classical Skolem set and the corresponding sequence for a k**-extended classical Skolem sequence**.

When $k = n+1$, $n = |A|$, the $n+1$-extended classical Skolem sequence is often referred to as a **Rosa sequence**. It is known due to Rosa [12] that a Rosa sequence gives a solution to Heffter's second difference problem. Just like classical Skolem sequences, k-extended classical Skolem sequences have been useful in several branches of mathematics like graph theory and design theory. See [3] for an example. Necessary and sufficient conditions for the existence of k-extended classical Skolem sequences were conjectured by Billington [2]. This conjecture was later proved by Baker [1].

Theorem 6.1 ([1]) *Sets of the form $k - \{1, 2, \ldots, n-1, n\}$ are Skolem sets if and only if $n \equiv 0, 1 \ (mod\ 4)$ for k odd and $n \equiv 2, 3 \ (mod\ 4)$ for k even.*

We give necessary conditions for the existence of k-extended Skolem sets.

Theorem 6.2 *A necessary condition for a set $k - A$ to be a Skolem set is that if k is odd, the number of even elements in A must be even. And if k is even, the number of even elements in A must be odd.*

Proof: This an almost immediate consequence of Theorem 2.1. Just note that if $k-A$ is a Skolem set then $A \cup k$ is a multi Skolem set. We know from Theorem 2.1 that a necessary condition for a multi set B to be a multi Skolem set is that it has an even number of even elements. Thus if k is odd, the number of even elements in A must be even, and if k is even, the number of even elements in A must be odd.

□

Before stating the second necessary condition, we introduce a very useful notation due to Iverson. The idea is simply to enclose a true or false statement in brackets, and to say that the result is 1 if the statement is true, 0 if it is false. For example,

$$[p \text{ prime}] = \begin{cases} 1, & \text{if } p \text{ is a prime number;} \\ 0, & \text{if } p \text{ is not a prime number.} \end{cases}$$

Theorem 6.3 *A necessary condition for a set $k - A$ to be a Skolem set is that it is dense enough. Let $n = |A|$, also assume that the elements in A are sorted in decreasing order. That is, $A = (a_1, a_2 \ldots a_n)$ where $a_i > a_j$ when $i < j$. Because of symmetry (remember that we can read a sequence both from right to left and from left to right) we need only to consider $k - A$ where $k \leq n+1$. A is dense enough if and only if the following inequality is satisfied for all $m \leq n$:*

$$\sum_{i=1}^{m} a_i \leq (2n-m)m + min\{k-1, m\}$$

Proof: This proof is very similar to the proof of Theorem 2.2. Given a k-extended Skolem sequence, we think of the numbers between a pair as fillers to that pair. We put the number of times a number appears as a filler in the exponent to that number. For example $8^0 0^1 3^1 4^2 2^3 3^3 2^2 4^1 8^0$, the first occurrence of 3 is only filler to the pair of eights, therefore it is given the exponent 1 and so on. The maximum sequence of exponents in a sequence generated from a set $k - A$ with $|A| = n$ and $k < n+1$ is: $0, 1, 2, \ldots, k-1, k-1, \ldots, n-1, n-1, n-2, \ldots, 2, 1, 0$. Similarly if $k = n$, the maximum sequence of exponents is $0, 1, 2, \ldots, n-1, n, n-1, n-2, \ldots, 2, 1, 0$. Note that by maximum sequence we mean that it has the largest possible value at all positions. Now look at the m largest elements in A, they can together take at most $\sum_{i=1}^{m} 2(n-i+1/2)[i < k] + 2(n-i)[i \geq k]$ elements as fillers (according to the maximum sequence of exponents, also keep in mind that no number can be placed in position k). The sum $\sum_{i=1}^{m}(a_i - 1)$ can be interpreted as the number of fillers required by the m largest elements in A. Clearly a set can not be a k-extended Skolem set if the m largest elements requires more fillers then what is maximum available. Thus we have the following inequality:

$$\sum_{i=1}^{m}(a_i - 1) \leq \sum_{i=1}^{m}(2(n-i+1/2)[i<k] + 2(n-i)[i \geq k])$$

Which is equivalent to:

$$\sum_{i=1}^{m} a_i \leq \sum_{i=1}^{m}(2(n-i+1)[i<k] + 2(n-i+1/2)[i \geq k])$$

If we write the sum on the right in closed form we get:

$$\sum_{i=1}^{m} a_i \leq (2n-m)m + min\{k-1, m\}$$

□

These necessary conditions are not sufficient. For example the set $3 - \{1, 3\}$ satisfies both conditions, but is not a Skolem set.

7 Generalized multi Skolem sequences

Generalized Skolem sequences was introduced in [1] where they are used to construct k-extended classical Skolem sequences. They have also been used in the construction of Steiner triple systems with prescribed automorphism groups and in the construction of extended Langford sequences with small defects, see [9]. A **generalized Skolem sequence** is a sequence of positive integers and null symbols such that an integer appears exactly twice or not at all, and the two appearances of an integer j are j positions apart. If the integers in A can be used to construct a generalized Skolem sequence using only the positions in P, we say that $P - A$ is a **generalized Skolem set**. In the spirit of our previous work we generalize the notion of generalized Skolem sequences slightly and allow the set A to be a multiset. We call these generalized sequences for **generalized multi Skolem sequences**, and the corresponding sets $P - A$ for **generalized multi Skolem sets**. For example, $\{1,2,4,5,7,8\} - \{1,6,6\}$ is a generalized multi Skolem set. The corresponding generalized multi Skolem sequence is 66011066 (the null symbol, 0, occupies positions that are not in P). We will prove that generalized Skolem sequences are $\mathcal{NP}$-complete by a reduction from the $\mathcal{NP}$-complete problem Perfect Graceful Matching (see Theorem 5.1).

GENERALIZED MULTI SKOLEM SEQUENCES
INSTANCE: A multiset A of positive integers, $|A| = m$, a set P of positive integers, $|P| = 2m$.
QUESTION: Is $P - A$ a generalized multi Skolem set? That is, can a generalized multi Skolem sequence be constructed from the elements in A, using only the positions in P?

Before presenting the proof we need to prove some properties of Perfect Graceful Matching that we will make use of in the proof to follow. Perfect Graceful Matching is $\mathcal{NP}$-complete even if no more than two edges have the same edge label. We know from Theorem 5.1 that Perfect Graceful Matching is $\mathcal{NP}$-complete even if no more than three edges have the same edge label. Thus we need to consider the components of the graph that have three edges with the same label and transform it into an equivalent component that has a perfect graceful matching if and only if the original component has one. This transformation is carried out in Figure 9. This transformation can clearly be done in polynomial time, and thus Perfect Graceful Matching is $\mathcal{NP}$-complete even if no more than two edges have the same edge label.

Another property of Perfect Graceful Matching that we need is that it is $\mathcal{NP}$-complete even if none of the edges with the same edge label share a common vertex. This is trivially true, see Figure 10.

To simplify the reduction we prove one final property of Perfect Graceful Matching. Perfect Graceful Matching is $\mathcal{NP}$-complete even if the number of different edge labels in the graph is greater than half the number of vertices in the graph. If m equals the number of different edge labels in the graph and the number of vertices is $2n$, (assuming that $m \leq n$) we add $(n - m) + 1$ copies of the component in Figure 11 to the graph. The resulting graph will have the property that $m > n$, and again it is easy to see that the resulting graph has a perfect graceful matching if and only if the original graph has one.

Theorem 7.1 Generalized Multi Skolem Sequences *is $\mathcal{NP}$-complete.*

Proof: The proof will be a reduction from the $\mathcal{NP}$-complete problem Perfect Graceful Matching. Because of the discussion above we need only to consider instances of Perfect Graceful Matching that have the following

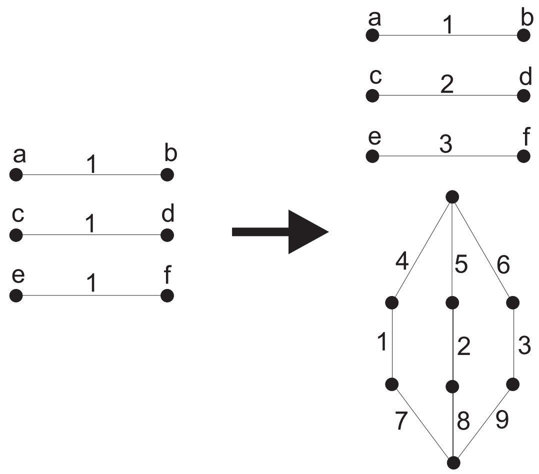

Figure 9: Edge labels occurring on the right hand side and not occurring on the left hand side (labels 2-9) are new ones not previously used in the graph.

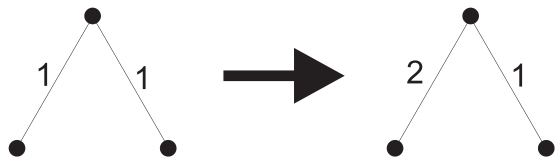

Figure 10: The label 2 are a new one not previously used in the graph.

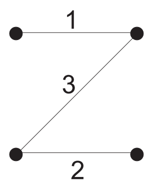

Figure 11: All edge labels (in each of the $(n-m)+1$ copies of the component above) are new ones not previously used in the graph.

properties:

1. No more than two edges have the same edge label.

2. No two edges with the same edge label share a common vertex.

3. The number of different edge labels in the graph is greater than half the number of vertices in the graph.

First note that it is easy to see that Generalized Multi Skolem Sequences $\in \mathcal{NP}$.

Overall structure: Given an instance of Perfect Graceful Matching, $G = (V, E)$ we will construct a set $P - A$ that is a generalized multi Skolem set if and only if G has a perfect graceful matching. The construction is quite complicated and will be divided into two main parts, the construction of a matching component and the construction of a garbage component. The matching component guarantees that if G does not have a perfect graceful matching, $P - A$ will not be a generalized multi Skolem set. The garbage component guarantees that if G has a perfect graceful matching, $P - A$ will be a generalized multi Skolem set. P can be represented as a sequence of ones and zeros, $p_i \in P$ if and only if position p_i in the sequence is occupied by a one. The polynomial under each component is the number of positions in that component, $n = |V|/2$, $m =$ the number of different edge labels in G. The overall structure of P is:

$$\underbrace{\text{matching component}\ldots}_{\substack{6nm^2+14nm+\\6m^2+2n+2m}}\ \underbrace{\text{barrier}\ldots}_{\substack{6nm^2+14nm+\\6m^2+2n+2m}}\ \underbrace{\text{garbage component}\ldots}_{\substack{18nm^3+36nm^2-6n^2m^2+\\8nm-14n^2m+18m^3+4m^2-4n^2}}$$

The overall structure of the matching component is:

$$\underbrace{\text{start pos}\ldots}_{\substack{2nm^2+2nm+\\2m^2}}\underbrace{\text{barrier}\ldots}_{\substack{2nm^2+8nm+\\2m^2}}\underbrace{\text{match pos}\ldots}_{2n}\underbrace{\text{barrier}\ldots}_{\substack{2nm^2+2nm+\\2m^2}}\underbrace{\text{stop pos}\ldots}_{2nm+2m}$$

Start pos consists of a sequence of m copies of the following component:

$$\underbrace{\text{start}\ldots}_{2n}\underbrace{\text{barrier}\ldots}_{2nm+2m}$$

Stop pos consists of a sequence of m copies of the following component:

$$\underbrace{\text{stop}\ldots}_{2n}\underbrace{\text{barrier}}_{2}$$

The overall structure of the garbage component is:

$$\underbrace{\text{garbage I}\ldots}_{\substack{12n^2m^2+28n^2m+\\12nm^2+4nm+4n^2}}\ \underbrace{\text{garbage II}\ldots}_{\substack{(m-n)(10nm^2+24nm+\\10m^2+2m+4n)}}\ \underbrace{\text{garbage III}\ldots}_{\substack{(m-n)(8nm^2+18nm+\\8m^2+2m+4n)}}$$

Garbage I consists of a sequence of n copies of the following component:

$$\underbrace{\text{start}}_{1}\ \underbrace{\text{barrier}\ldots}_{\substack{4nm^2+10nm+\\4m^2+2n}}\ \underbrace{\text{absorb}\ldots}_{\substack{2nm^2+4nm+\\2m^2+2m-1}}\ \underbrace{\text{barrier}\ldots}_{\substack{6nm^2+14nm+\\6m^2+2m+2n}}$$

Garbage II consists of a sequence of $m - n$ copies of the following component:

$$\underbrace{\text{start}}_{1}\underbrace{\text{barrier}\ldots}_{\substack{2nm^2+\\8nm+2m^2}}\underbrace{\text{absorb}\ldots}_{\substack{2nm^2+2nm+\\2m^2+2n-1}}\ \underbrace{\text{barrier}\ldots}_{\substack{6nm^2+14nm+\\6m^2+2m+2n}}$$

Garbage III consists of a sequence of $m-n$ copies of the following component:

$$\underbrace{\text{start}}_{1}\underbrace{\text{barrier}\ldots}_{\substack{2nm^2+\\2nm+2m^2}}\underbrace{\text{absorb}\ldots}_{\substack{2nm+\\2m+2n-1}}\ \underbrace{\text{barrier}\ldots}_{\substack{6nm^2+14nm+\\6m^2+2m+2n}}$$

Assume that the vertices in the graph G have been labeled with the numbers $1\ldots 2n$, and that the edges have been labeled with the numbers $1\ldots m$. Also assume that if (i,j) is an edge in G, then $i<j$. All positions in the barrier components are occupied by zeros, thus they are not in P. P and A will be constructed in a step by step fashion, starting with the matching component. In each step we label the set constructed there in the most obvious way. For example if we want to add a set of elements to A in Step M2.1, we denote this set $A_{M2.1}$. Of course we then let A be the union of all these sets, that is $A=A_x\cup A_y\cup\ldots\cup A_z$. P is handled in exactly the same way. We begin with giving the precise construction of P and A in a very formal way. A more easy to understand explanation of what is actually achieved in every step is then given. Also keep in mind that A is a multiset, i.e. $A=\{1,2,3\}\cup\{1,3\}=\{1,1,2,3,3\}$.

Matching component

Step M1: This step deals with the positions in match pos.
Step M1.1: Add all positions in match pos to P, that is $P_{M1.1}=\{4nm^2+10nm+4m^2+1,\ldots,4nm^2+10nm+4m^2+2n\}$.

Step M2: This step deals with the positions in start pos and stop pos.
Step M2.1: For all $k\in 0\ldots m-1$, let $A_{M2.1.k}=\{4nm^2+10nm+4m^2-k(2nm+2n+2m)\}$
Step M2.2: For all $k\in 0\ldots m-1$, let $A_{M2.2.k}=\{2nm^2+4nm+2m^2+2m+2n-(k+1)(2n+2)\}$.
Step M2.3: For each edge (i,j) in G labeled k, $k\in 0\ldots m-1$, let $P_{M2.3.k.(i,j)}=\{k(2nm+2n+2m)+i, 6nm^2+14nm+6m^2+2m+2n-(k+1)(2n+2)+j\}$.
Step M2.4: For each edge (i,j) in G labeled k, $k\in 0\ldots m-1$, let $A_{M2.4.k.(i,j)}=\{(6nm^2+14nm+6m^2+2m+2n-(k+1)(2n+2)+j)-(k(2nm+2n+2m)+i)\}$.

Garbage component:

Step G1: This step deals with the positions in garbage I.
Step G1.1: Remember that the garbage I component consists of a sequence of n identical components. In each of these n components, add the position in start and all positions in absorb to P. That is for all $k\in 0\ldots n-1$, let $P_{G1.1.k}=\{(k+1)(12nm^2+28nm+12m^2+4n+4m)+1\}\cup\{(k+1)(12nm^2+28nm+12m^2+4n+4m)+4nm^2+10nm+4m^2+2n+2,\ldots,(k+1)(12nm^2+28nm+12m^2+4n+4m)+6nm^2+14nm+6m^2+2n+2m\}$
Step G1.2: Let $A_{G1.2}=\{1^{(n^2m^2+2n^2m+m^2+m-3n)},2^{2n}\}$ (Note that a^b means b copies of a).

Step G2: This step deals with the positions in garbage II.
Step G2.1: Remember that the garbage II component consists of a sequence of $m-n$ identical components. In each of these $m-n$ components, add the position in start and all positions in absorb to P. That is for all $k \in 0 \ldots (m-n)-1$, let $P_{G2.1.k} = \{12n^2m^2+28n^2m+24nm^2+32nm+4n^2+12m^2+4n+4m+k(10nm^2+24nm+10m^2+2m+4n)+1\} \cup \{12n^2m^2+28n^2m+26nm^2+40nm+4n^2+14m^2+4n+4m+k(10nm^2+24nm+10m^2+2m+4n)+2, \ldots, 12n^2m^2+28n^2m+28nm^2+42nm+4n^2+16m^2+6n+4m+k(10nm^2+24nm+10m^2+2m+4n)\}$
Step G2.2: Let $A_{G2.2} = \{1^{(m-n)(nm^2+nm+m^2+n-2)}, 2^{(m-n)}\}$.

Step G3: This step deals with the positions in garbage III.
Step G3.1: Remember that the garbage III component consists of a sequence of $m-n$ identical components. In each of these $m-n$ components, add the position in start and all positions in absorb to P. That is for all $k \in 0 \ldots (m-n)-1$, let $P_{G3.1.k} = \{2n^2m^2+4n^2m+10nm^3+38nm^2+34nm+10m^3+14m^2+4n+4m+k(8nm^2+18nm+8m^2+2m+4n)+1\} \cup \{2n^2m^2+4n^2m+10nm^3+40nm^2+36nm+10m^3+16m^2+4n+4m+k(8nm^2+18nm+8m^2+2m+4n)+2, \ldots, 2n^2m^2+4n^2m+10nm^3+40nm^2+38nm+10m^3+16m^2+6n+6m+k(8nm^2+18nm+8m^2+2m+4n)\}$

Step G3.2: Let $A_{G3.2} = \{1^{(m-n)(nm+n+m-2)}, 2^{(m-n)}\}$.

Explanation

Step M1: Here (in Step M1.1) we add all $2n$ positions in match pos to P. Match pos is in some sense the heart of the whole construction. The $2n$ positions in match pos represents the $2n$ vertices in G. The first position in match pos represents the vertex labeled 1, the second position represents the vertex labeled 2, and so on.

Step M2: Here we add some elements to A, and certain positions in start pos and stop pos to P. Remember that start pos and stop pos each consists of m blocks, one for each of the different edge labels in G. The k:th leftmost block in start pos and the k:th rightmost block in stop pos is dealing with the edges in G labeled k.

In Step M2.1 we add one element to A for each of the m different edge labels in G, these elements are used to represent the different edge labels in G and transfer information from the start pos component to the match pos component. More specifically, the element in $A_{M2.1.k}$ is used to transfer information from the k:th leftmost block in start pos to the match pos component. Remember that if $a \in A$, a must be placed at positions p_i and p_j in the sequence, such that $|p_i - p_j| = a$ and $\{p_i, p_j\} \subseteq P$. The structure of P makes sure that the element in $A_{M2.1.k}$ can only be fitted in the matching component if one of the copies is placed in match pos, and the other one is placed in the k:th leftmost block in start pos. The element in $A_{M2.1.k}$ have the additional property that if one of the copies is placed at the i:th leftmost position in its corresponding start pos block, the other copy must be placed at the i:th leftmost position in match pos. This is how we in some sense can transfer information from the start pos component to the match pos component. Note that if an element in A is not fitted in the matching component it must be fitted in the garbage component.

The elements added to A in Step M2.2 have the same function and properties as the elements added to A in Step M2.1, with the difference that they transfer information from the stop pos component to the match pos component. Also note that the element in $A_{M2.2.k}$, is used to transfer information from the k:th rightmost block (as opposed to from the k:th leftmost block, as were the case for the element in $A_{M2.1.k}$) in stop pos to the match pos component.

In Step M2.3 we do the following, for each edge (i, j) in G labeled k, $k \in 0 \ldots m-1$, we add the i:th leftmost position in the k:th leftmost block in start pos, and the j:th leftmost position in the k:th rightmost block in stop pos to P. All other positions in start pos and stop pos are occupied by zeros (are not in P). Thus we have made sure that an element from A can be fitted with one copy at the i:th leftmost position in the k:th leftmost block in start pos and the other copy at the i:th leftmost position in match pos if and only if there is an edge in G labeled k starting at vertex i. Similarly, an element from A can be fitted with one copy at the j:th leftmost position in the k:th rightmost component in stop pos and the other copy at the j:th leftmost position in match pos if and only if there is an edge in G labeled k ending at vertex j.

Now let us take a moment and think of what remains to be done. We must make sure that either both of the two elements in $A_{M2.1.k}$ and $A_{M2.2.k}$ are fitted in the garbage component, or both of them are fitted in the matching component. We also must make sure that if the element in $A_{M2.1.k}$ is fitted with one copy at the i:th leftmost position in match pos and the element in $A_{M2.2.k}$ is fitted with one copy at the j:th leftmost position in match pos, then (i, j) is an edge (labeled k) in G. This is exactly what is achieved in Step M2.4. We formulate this as a lemma.

Lemma 7.2 *Either both of the two elements in $A_{M2.1.k}$ and $A_{M2.2.k}$ are fitted in the garbage component, or both of them are fitted in the matching component. And if the element in $A_{M2.1.k}$ is fitted with one copy at the i:th leftmost position in match pos and the element in $A_{M2.2.k}$ is fitted with one copy at the j:th leftmost position in match pos, then (i, j) is an edge (labeled k) in G.*

Proof: In Step M2.4 we add one element to A for every edge (i, j) in G. This element can only be fitted in the matching component if one of the copies is placed at the i:th leftmost position in the k:th leftmost block in start pos, and the other copy is placed at the j:th leftmost position in the k:th rightmost block in stop pos (under the assumption that the edge (i, j) is labeled k). This is where we make use of the fact that at most two edges in G have the same edge label. If more than two edges would have the same edge label, the above statement would not hold. In fact the above statement need not hold anyway, consider the case where both the edges (i, j) and $(i + w, j + w)$ in G are labeled

k. Then the element in $A_{M2.4.k.(i,j)}$ could be fitted in the matching component in the same place where the element in $A_{M2.4.k.(i+w,j+w)}$ would fit. This is of course harmless since the two elements added to A for these edges would be identical, thus nothing bad could happen if they were swapped.

Now assume that the element in $A_{M2.1.k}$ is fitted with one copy at the i:th leftmost position in match pos and the element in $A_{M2.2.k}$ is fitted with one copy at the j:th leftmost position in match pos, but that (i,j) is not an edge (labeled k) in G. This could only happen if there were two edges, (i,x) and (y,j) labeled k in G. But now consider the y:th leftmost position in the k:th leftmost block in start pos. This position is empty and must be filled by an element in A, but no such element exists. The only two candidates were the element already used at the i:th leftmost position in match pos, and the element in $A_{M2.4.k.(y,j)}$ (this can not be used either because the j:th leftmost position in the k:th rightmost block is already taken).

Now assume that just one of the two elements in $A_{M2.1.k}$ and $A_{M2.2.k}$ is placed in the matching component. But following the same line of reasoning as above there would be no element in A to fit in the empty position (in the k:th leftmost/rightmost block in start pos/stop pos) where the missing element should have been placed.

We have thus made sure that either both of the two elements in $A_{M2.1.k}$ and $A_{M2.2.k}$ are fitted in the garbage component, or both of them are fitted in the matching component. We have also made sure that if the element in $A_{M2.1.k}$ is fitted with one copy at the i:th leftmost position in match pos and the element in $A_{M2.2.k}$ is fitted with one copy at the j:th leftmost position in match pos, then (i,j) is an edge (labeled k) in G.

□

Now we are in a position to prove that if G does not have a perfect graceful matching then the constructed $P-A$ can not be a generalized multi Skolem set. But first we need a few facts about the garbage component. It is easy to see that exactly n of the elements added to A in Step M2.4 must be fitted in the garbage component, more specifically the garbage I component (because no other elements in A can be put in the start position of the garbage I component). Similarly it is easy to see that exactly $(m-n)$ of the elements added to A in Step M2.1 must be fitted in the garbage component, more specifically the garbage II component.

Given a graph G that have no perfect graceful matching, assume that the constructed $P-A$ is a generalized multi Skolem set. We will show that this is impossible, because of the structure of $P-A$. The remaining n elements added to A in Step M2.1 must be fitted in the matching component, because they simply do not fit anywhere else. And since we know from Lemma 7.2 that either both of the two elements in $A_{M2.1.k}$ and $A_{M2.2.k}$ must be fitted in the garbage component, or both of them must be fitted in the matching component, we can conclude that exactly n of the elements added to A in Step M2.2 must be fitted in the matching component.

Now consider a specific generalized multi Skolem sequence generated from $P-A$ (such a sequence must exist since we have assumed $P-A$ to be generalized multi Skolem set). Lemma 7.2 says that if the element in $A_{M2.1.k}$ is fitted with one copy at the i:th leftmost position in match pos and the element in $A_{M2.2.k}$ is fitted with one copy at the j:th leftmost position in match pos, then (i,j) is an edge (labeled k) in G. So for all such elements in the sequence, let the edge $(i,j) \in M$. Then clearly M will be a perfect graceful matching on G. This is a contradiction and our assumption that $P-A$ was a generalized multi Skolem set was wrong. And thus we have proved that if G does not have a perfect graceful matching then the constructed $P-A$ can not be a generalized multi Skolem set.

Now what remains to be proved is that if G has a perfect graceful matching M then the constructed $P-A$ is a generalized multi Skolem set. This is not too complicated. Remember that every element added to A in Step M2.4 corresponds to a specific edge (i,j) in G. If this edge $(i,j) \notin M$, then fit the element in $A_{M2.4.k.(i,j)}$ in the matching component (it can only be fitted in one place in the matching component). Note that this is where we make use of the fact that no two edges in G of the same edge label share a common vertex. Because if this property was not true we could not guarantee that this element could be fitted in the matching component at all. For all k such that M contains an edge (i,j) labeled k, fit the element in $A_{M2.1.k}$ with one copy at the i:th leftmost position in match pos, and fit the element in $A_{M2.2.k}$ with one copy at the j:th leftmost position in match pos. Now we have filled all positions in P in the matching component. But some of the elements added to A in Step M2.1, Step M2.2 and Step M2.4 have not been fitted yet. More specifically the elements that remain to be fitted are $m-n$ of the elements added to A in Step M2.1, $m-n$ of the elements added to A in Step M2.2 and n of the elements added to A in Step M2.4. These superfluous elements must be fitted in the garbage component. How this is done is explained in Step G1-3 below.

Step G1: Here we make sure that the n superfluous elements added to A in Step M2.4 can be fitted in the garbage component. We know that there will be exactly n of them because the elements added to A in Step M2.4 are fitted in the matching component if and only if the corresponding edge in G is not part of the perfect graceful matching. We also know that an even number of these n elements are even. This is because the element in $A_{M2.4.k.(i,j)}$ is of the form $(2x+j)-(2y+i)$ which has the same parity as $j-i$. We can view the perfect graceful matching in G as a multi Skolem sequence in the usual way. The n superfluous elements are exactly those that were added to A for the n edges in the perfect graceful matching. And because it is a necessary condition for the existence of multi Skolem sequences that the number of even elements are even (see Theorem 2.1), we know that an even number of these n superfluous elements are even.

In Step G1.1 we add all positions in start and absorb in each of the n identical blocks in garbage I to P.

In Step G1.2 we add $n^2m^2 + 2n^2m + m^2 + m - 3n$ ones and $2n$ twos to

A. What we want to show is that the n superfluous elements added to A in Step M2.4 and the elements added to A in Step G1.2 can always be fitted in the garbage I component. First note that the number of positions in P in the garbage I component is exactly twice the number of elements from A that we want to fit here, thus all positions in P will be filled. Place one element of the n superfluous elements added to A in Step M2.4 in each of the n blocks in garbage I, such that the first copy is placed at the start position and the second copy is placed somewhere in absorb. If the element that was placed is even, place a two at the two positions that are neighbors to the position in absorb where the second copy was placed. This guarantees that the positions in P that remain to be filled, are divided into blocks of consecutive positions of even length. These positions can clearly be filled by just placing ones at these, but we have too few ones, and some twos also remain to be placed. Remember that at the beginning we had $2n$ twos to place, we have placed an even number of these, thus an even number of twos remain to be placed. This fact and the observation that 1111 can be replaced by 2222 makes sure that we can fill these remaining positions in P, by using the remaining elements that where added to A in Step G1.2. Thus we can conclude that the n superfluous elements added to A in Step M2.4 and the elements added to A in Step G1.2 can always be fitted in the garbage I component. Of course leaving no positions in P in the garbage I component unfilled.

Step G2: Here we make sure that the $m - n$ superfluous elements added to A in Step M2.1 can be fitted in the garbage component. We know that there will be exactly $m - n$ of them because the elements added to A in Step M2.1 are fitted in the matching component if and only if the corresponding edge label in G is part of the perfect graceful matching. Since we have m different edge labels and only n of these can be part of the perfect graceful matching, this leaves $m - n$ elements in A that can not be fitted in the matching component. Note that all the elements added to A in Step M2.1 are even.

In Step G2.1 we add all positions in start and absorb in each of the $m - n$ identical blocks in garbage II to P.

In Step G2.2 we add $(m - n)(nm^2 + nm + m^2 + n - 2)$ ones and $m - n$ twos to A. What we want to show is that the $m - n$ superfluous elements added to A in Step M2.1 and the elements added to A in Step G2.2 can always be fitted in the garbage II component. First note that the number of positions in P in the garbage II component is exactly twice the number of elements from A that we want to fit here, thus all positions in P will be filled. Place one element of the $m - n$ superfluous elements added to A in Step M2.1 in each of the $m - n$ blocks in garbage II, such that the first copy is placed at the start position and the second copy is placed somewhere in absorb. Since all these elements that was placed are even, place a two at the two positions that are neighbors to the position in absorb where the second copy was placed. This guarantees that the positions in P that remain to be filled, are divided into blocks of consecutive positions of even length. These positions can clearly be filled by just placing the ones added to A in Step G2.2 there. Thus we can conclude that the $m - n$ superfluous elements added to A in Step M2.1 and the elements added to A in

Step G2.2 can always be fitted in the garbage II component. Of course leaving no positions in P in the garbage II component unfilled.

Step G3: Here we make sure that the $m-n$ superfluous elements added to A in Step M2.2 can be fitted in the garbage component. We know that there will be exactly $m-n$ of them because the elements added to A in Step M2.2 are fitted in the matching component if and only if the corresponding edge label in G is part of the perfect graceful matching. Since we have m different edge labels and only n of these can be part of the perfect graceful matching, this leaves $m-n$ elements in A that can not be fitted in the matching component. Note that all the elements added to A in Step M2.2 are even.

In Step G3.1 we add all positions in start and absorb in each of the $m-n$ identical blocks in garbage III to P.

In Step G3.2 we add $(m-n)(nm+n+m-2)$ ones and $m-n$ twos to A. The garbage III component works in exactly the same way as the garbage II component, so we can conclude without further discussion that the $m-n$ superfluous elements added to A in Step M2.2 and the elements added to A in Step G3.2 can always be fitted in the garbage III component. Of course leaving no positions in P in the garbage III component unfilled.

This shows that the constructed $P-A$ is a generalized multi Skolem set. Thus we have proved that if G has a perfect graceful matching M then the constructed $P-A$ is a generalized multi Skolem set.

It is easy to see that the reduction can be done in polynomial time, just verify that every step above can be done in polynomial time. After the detailed explanation above it should be clear that the constructed $P-A$, is a generalized multi Skolem set if and only if G has a perfect graceful matching. Thus we have proved that GENERALIZED MULTI SKOLEM SEQUENCES is $\mathcal{NP}$-complete.

$\square$

An actual example of the reduction in action might have been nice to enhance the understanding of the reduction. But unfortunately even a small graph with only six vertices and four edges is reduced to a sequence of 6224 positions.

8 Restricted involutions with a given signature

We define the signature of an involution I in S_n to be a sequence $(x_0, x_1, \ldots, x_{n-1})$, such that $x_0 =$ the number of fixed points in I, $x_1 =$ the number of transpositions (a, b) such that $|a-b| = 1$, and so on. For example the involution $(1,3)(4,5)$ in S_5 has the signature $(1,1,1,0,0)$.

RESTRICTED INVOLUTIONS WITH A GIVEN SIGNATURE
INSTANCE: A $n \times n$ matrix M with some positions forbidden and a signature $(x_0, x_1 \ldots x_i)$.

QUESTION: Can an involution with the given signature be fitted in the matrix (without using any forbidden positions).

See Figure 12 for an example.

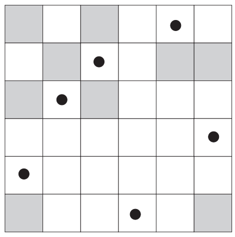

Figure 12: An instance of Restricted Involutions With A Given Signature, the shaded positions in the matrix are forbidden, the signature is $(0, 1, 1, 0, 1, 0)$. The provided involution is $(1, 5)(2, 3)(4, 6)$.

Theorem 8.1 Restricted Involutions With A Given Signature *is* $\mathcal{NP}$*-complete.*

Proof: This is a direct consequence of the $\mathcal{NP}$-completeness of Generalized Multi Skolem Sequences. First of all it is easy to see that Restricted Involutions With A Given Signature $\in \mathcal{NP}$. The reduction from Generalized Multi Skolem Sequences to Restricted Involutions With A Given Signature is fairly straight forward, at least when compared to the $\mathcal{NP}$-completeness proof of Generalized Multi Skolem Sequences.

Given an instance $P - A$ of Generalized Multi Skolem Sequences, find the largest element n in P and create a $n \times n$ matrix. For all elements $x \in P$, let (x, x) be a forbidden position. For all elements $y < n$ such that $y \notin P$, let all positions in row y and column y, except position (y, y), in M be forbidden.

Now on to the signature, $(x_0, x_1, \ldots, x_{n-1})$. Remember that x_0 is the number of fixed points. Let $x_0 = n - |P|$. For all $k \in \{1 \ldots n-1\}$, let $x_k =$ the number of k:s in A.

This transformation can be done in polynomial time and it is easy to see that the constructed instance of Restricted Involutions With A Given Signature has a solution if and only if $P - A$ is a generalized multi Skolem set. Thus we have proved that Restricted Involutions With A Given Signature is $\mathcal{NP}$-complete. For a clarifying example of the reduction, see Figure 13.

$\square$

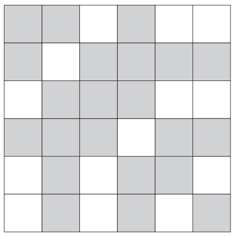

Figure 13: The Generalized Multi Skolem Sequences instance $\{1,3,5,6\} - \{2,5\}$ is transformed into the above matrix and the following signature $(2,0,1,0,0,1)$. $\{1,3,5,6\} - \{2,5\}$ is a generalized multi Skolem set (solution: 502025), thus the constructed instance of Restricted Involutions With A Given Signature has a solution, namely $(1,6)(3,5)$.

9 Skolem permutations

As we have noted before, a Skolem sequence of order n can be seen as a fixed point free involution in S_{2n}, where the distances between the transposed elements are distinct. A classical Skolem sequence of order n can be seen as a fixed point free involution σ in S_{2n}, where the distances between the transposed elements are $\{1,2,\ldots,n\}$. Another way to express this is to say that for all $k \in \{1,2,\ldots,n\}$ there exist two elements i,j such that $|i-\sigma(i)| = |j-\sigma(j)| = k$, $\sigma(i) = j$ and $\sigma(j) = i$. The result due to Skolem [18] tells us that such an involution exists if and only if $n \equiv 0$ or 1 $(mod\ 4)$. Now if we turn to permutations instead of involutions, the problem that immediately suggests itself is this, for which n do there exist a permutation π in S_n such that for all $k \in \{0,1,\ldots,n-1\}$ there exist an element i such that $|i-\pi(i)| = k$. The problem can be restated as follows, for which n do there exist a permutation π in S_n with the property that no two elements j,k such that $|j-\pi(j)| = |k-\pi(k)|$ exist? We call these permutations **Skolem permutations**. The existence question for such permutations was also recently raised by ShahAli in the problems section of Math. Monthly [16]. We will give a complete solution to the problem.

Theorem 9.1 *A necessary and sufficient condition for the existence of a Skolem permutation π_S in S_n is that $n \equiv 0$ or 1 $(mod\ 4)$.*

Proof: We begin by proving the necessity of the condition. We say that an element i is at its origin if $i - \pi(i) = 0$, thus the identity permutation is the permutation where all elements are at their origin. Define the function ϕ to be the distance between a permutation π in S_n and the identity permutation. That is $\phi = \sum_{i=1}^{n}(|i-\pi(i)|)$. For example, if $\pi = \binom{1234}{2431}$ then $\phi = 1+2+0+3 = 6$. If we let two adjacent elements in the permutation be transposed, one of the following three cases occurs:

1. Both elements are taken one step closer to their origin, ϕ is therefore

decreased by 2.

2. Both elements are taken one step further away from their origin, ϕ is therefore increased by 2.

3. One of the elements is taken one step closer to its origin, while the other element is taken one step further away from its origin, ϕ remains unchanged.

We see that the parity of ϕ remains unchanged when we transpose adjacent elements in the permutation. It is also easy to see that given a permutation π in S_n, we can construct all possible permutations in S_n by repeated transposition of adjacent elements. Clearly we can construct the identity permutation, ϕ is then 0 (even parity). Thus it follows that every permutation must have a ϕ that is even. Since ϕ of the Skolem permutation $\pi_s = 0+1+\ldots+(n-1) = n(n-1)/2$ it is easy to see that $n \equiv 0$ or 1 $(mod\ 4)$ is a necessary condition for such a permutation to exist.

The sufficiency of the condition follows neatly from a theorem in graph theory. A function f is called a graceful labeling of a graph G with n edges if f is an injection from the vertices of G to the set $\{1, 2, \ldots, n+1\}$ such that, when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct. A graph G that has such a graceful labeling f is called graceful. For a survey on graceful labeling see [4]. Rosa showed that the n-cycle C_n is graceful if and only if $n \equiv 0$ or 3 $(mod\ 4)$ [13]. A graceful labeling of C_n can be seen as a permutation in S_{n+1} in the following way. Let π_s be the permutation where every element (vertex) in the gracefully labeled n-cycle is sent to its neighbor in clockwise direction. Note that since the n-cycle has n edges and n vertices, one of the elements in $\{1, 2, \ldots, n+1\}$ is not assigned to a vertex. Let this element be a fixed point of π_s, i.e. it is sent to itself. For a clarifying example of how a graceful labeling of C_n can be transformed into a Skolem permutation in S_{n+1}, consider the graceful graph C_4 where the vertices has been labeled (in clockwise direction) $1, 5, 3, 4$, then the corresponding Skolem permutation in S_5 is $\binom{12345}{52413}$. It should be clear that the permutation π_s in S_{n+1} has the property that no two elements j, k such that $|j - \pi_1(j)| = |k - \pi_1(k)|$ exist and thus is a Skolem permutation. It follows that the condition $n \equiv 0$ or 1 $(mod\ 4)$ is sufficient for the existence of a Skolem permutation π_s in S_n.

$\square$

Thus we have proved that there is a Skolem permutation π_s in S_n if and only if $n \equiv 0$ or 1 $(mod\ 4)$.

10 Conclusions

Although a lot of effort has been put into finding a proof of Conjecture 2.3, all attempts so far has been fruitless. Unfortunately such fruitless attempts tend to have little or no impact on the written material. In fact about 3/4 of the time that was spent on writing this thesis was devoted to finding a proof of the conjecture. But as it has been pointed out to me, even the special case of the conjecture where only two elements are missing from the set is unsettled (Conjecture 2.4). Even a proof of the conjecture for this special case would be a nice and publishable result.

The enumerative result on multi Skolem sequences (Theorem 3.1) was a bit of a surprise, especially considering the amount of effort that has been put into finding similar results about classical Skolem sequences and Langford sequences.

The $\mathcal{NP}$-completeness result of GENERALIZED MULTI SKOLEM SEQUENCES (Theorem 7.1) can be seen as an upper bound on how far the generalizations of Skolem sequences can be taken and still hope to resolve the existence question in a nice way.

Although the necessary and sufficient condition for the existence of Skolem permutations and classical Skolem sequences are identical, there seems to be no direct link between the two problems.

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