Perfect Skolem sets

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Abstract

A Skolem sequence is a sequence $s_1, s_2, \ldots, s_{2n}$ (where $s_i \in A = \{1 \ldots n\}$), each s_i occurs exactly twice in the sequence and the two occurrences are exactly s_i positions apart. A set A that can be used to construct Skolem sequences is called a Skolem set. The problem of deciding which sets of the form $A = \{1 \ldots n\}$ are Skolem sets was solved by Thoralf Skolem in the late 1950's. We study the natural generalization where A is allowed to be any set of n positive integers. We give necessary conditions for the existence of Skolem sets of this generalized form. We conjecture these necessary conditions to be sufficient, and give computational evidence in favor of our conjecture. We investigate special cases of the conjecture and prove that the conjecture holds for some of them. We also study enumerative questions and show that this problem has strong connections with problems related to permutation displacements.

Key words: Skolem sequence, permutation displacement, design theory

1 Introduction

Skolem sequences originates from the work by Thoralf Skolem in 1957 [15] on the construction of Steiner triple systems. He asked when you could partition the set $\{1, 2, \ldots, 2n\}$ in n pairs (s_i, t_i) such that $t_i - s_i = i$ for $i = 1, 2, \ldots, n$. Later, this problem was reformulated into the (equivalent) problem of deciding which sets $A = \{1, 2, \ldots, n\}$ could be used to form a sequence with two copies of every element k in A so that the two copies of k were placed k places apart in the sequence. For example, the set $\{1, 2, 3, 4\}$ can be used to form the sequence 42324311, but the set $\{1, 2, 3\}$ cannot be used to form such a sequence. Close

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relatives to Skolem sequences are Langford sequences. A *Langford sequence* is a Skolem sequence which starts late. That is, a sequence that is constructed from a set of the form $\{a, a+1, \ldots, b-1, b\}$. Dudley Langford published an article in 1958 [6] where he managed to construct such sequences from the sets $\{2, 3, \ldots, n\}$ with $n = 5, 6, 9$ and 10.

Since these early papers appeared many different aspects of Skolem sequences and Langford sequences have been studied. One reason for them being so well studied is that they have important applications in several branches of mathematics, most notably in design theory and graph labeling; see [13].

A natural generalization of Skolem sequences that is not very well studied is when we allow the set of integers used to generate the sequences to be any set or multiset of positive integers. We call such sequences *perfect Skolem-type sequences*. This generalization is the main topic of this article.

We call a set that can be used to construct a perfect Skolem-type sequence a *perfect Skolem set*. For example the set $\{2, 3, 5, 6\}$ is a perfect Skolem set since it can be used to generate the sequence 56232536. When we want to point out that the sequences are constructed from multisets we call the sequences *perfect multi Skolem-type sequences* and the corresponding multiset is a *perfect multi Skolem set*.

The problem of deciding whether the set $P = \{1, 2, \ldots, 2n\}$ can be partitioned into the differences in $A = \{a_1, a_2, \ldots, a_n\}$ is exactly the problem of deciding whether A is a perfect multi Skolem set. We can generalize the notion of Skolem sequences even further by allowing the set of positions P to be an arbitrary set. If the set of positions P can be partitioned into the differences A, then we say that (P, A) is *generalized multi Skolem* (if A is not allowed to be a multi set, then we say that (P, A) is *generalized Skolem*). For example, $P = \{1, 2, 4, 5, 7, 8\}$, $A = \{1, 6, 6\}$ is generalized multi Skolem by the sequence 66_11_66. Given a set P and a multiset A, the problem of deciding whether (P, A) is generalized multi Skolem is NP-complete [10].

We now state some of the most important known results that are related to the generalization that we are about to study.

Theorem 1 ([15]) *$\{1, 2, \ldots, n\}$ is a perfect Skolem set if and only if $n \equiv 0, 1 \pmod 4$.*

The preceding theorem classify those sets which can be used to construct Skolem sequences.

Theorem 2 ([4,14]) *A set $A = \{a, a+1, \ldots, b\}$ is a perfect Skolem set if and only if*

(1) $|A| \geq 2a - 1$*; and*
(2) $|A| \equiv 0, 1 \pmod 4$ *when* a *is odd,* $|A| \equiv 0, 3 \pmod 4$ *when* a *is even.*

The preceding theorem classify those sets which can be used to construct Langford sequences.

Theorem 3 ([12]) *A set* $\{1, 2, \ldots, m-1, m+1, \ldots, n\}$ *is a perfect Skolem set if and only if* $n \equiv 0, 1 \pmod 4$ *and* m *is odd, or* $n \equiv 2, 3 \pmod 4$ *and* m *is even.*

Sequences constructed from these sets in the preceding theorem are known in the literature as near-Skolem sequences.

Theorem 4 ([2]) *A multiset* $\{1^m, 2^m, \ldots, n^m\}$ *is a perfect multi Skolem set if and only if*

(1) $n \equiv 0, 1 \pmod 4$*; or*
(2) $n \equiv 2, 3 \pmod 4$ *and* m *is even.*

Note that by a^b we mean b copies of a. Sequences constructed from these sets in the preceding theorem are known in the literature as m-fold Skolem sequences.

Theorem 5 ([1]) $P = \{1, 2, \ldots, k-1, k+1, \ldots, 2n+1\}$ *and* $A = \{1, 2, \ldots, n\}$ *is generalized Skolem if and only if* $n \equiv 0, 1 \pmod 4$ *for* k *odd and* $n \equiv 2, 3 \pmod 4$ *for* k *even.*

Sequences constructed from these sets in the preceding theorem are known in the literature as k-extended Skolem sequences.

2 Necessary conditions and conjectures

We give necessary conditions for a set A to be a perfect Skolem set. These necessary conditions are then conjectured to be sufficient for a set A to be a perfect Skolem set. We first state the necessary conditions in a more general form.

Theorem 6 *Let* $A = \{a_1, a_2, \ldots, a_n\}$ *be a multiset of differences with* $a_1 \leq a_2 \leq \ldots \leq a_n$*, and let* $P = \{p_1, p_2, \ldots, p_{2n}\}$*, with* $p_1 < p_2 < \ldots < p_{2n}$*. If* P *can be partitioned into the differences in* A*, then*

$$\sum_{i=1}^{n} a_i \equiv \sum_{i=1}^{2n} p_i \pmod 2, \tag{1}$$

$$\sum_{i=m}^{n} a_i \leq \left(\sum_{i=n+m}^{2n} p_i \right) - \left(\sum_{i=1}^{n-m+1} p_i \right), \quad (m = 1, \ldots, n). \tag{2}$$

PROOF. Let $a_i = t_i - s_i$, $(i = 1, 2, \ldots, n)$. Then $a_i \equiv t_i + s_i \pmod 2$ and summing over i gives (1). Also for each $1 \leq m \leq n$ we have $a_m + \ldots + a_n = (\sum_{i=m}^{n} t_i) - (\sum_{i=m}^{n} s_i) \leq (\sum_{i=n+m}^{2n} p_i) - (\sum_{i=1}^{n-m+1} p_i)$, and (2) is proved. □

Conditions 1 and 2 are referred to as parity and density conditions respectively. Note that it is easy to see that these conditions are not sufficient for (P, A) to be generalized Skolem. For example $P = \{1, 2, 4, 5\}$ and $A = \{1, 3\}$ satisfies both 1 and 2 but is not generalized Skolem.

We leave it up to the reader to verify the following consequence of the preceding theorem.

Corollary 7 *If $P = \{1, 2, \ldots, 2n\}$, then the two necessary conditions in the preceding theorem reduce to (1) the number of even a_i's is even, and (2) $\sum_{i=m}^{n} a_i \leq n^2 - (m-1)^2$ for each $1 \leq m \leq n$.*

Surprisingly, when $P = \{1, 2, \ldots, 2n\}$ and A is an ordinary set (i.e., not a multiset), then the necessary conditions seem to be sufficient.

Conjecture 8 *A set $A = \{a_1, a_2, \ldots, a_n\}$ with $a_1 < a_2 < \ldots < a_n$ is a perfect Skolem set if and only if the number of even a_i's is even, and $\sum_{i=m}^{n} a_i \leq n^2 - (m-1)^2$ for each $1 \leq m \leq n$.*

The conjecture has been verified by computer search for all sets of cardinality 20 or less. The conjecture does not hold when we allow the sets to be multisets. A minimal counterexample is the multiset $\{1, 3, 3\}$, it passes both necessary conditions, but cannot be used to construct a perfect multi Skolem-type sequence. The existence results for Skolem sequences, Langford sequences, and near-Skolem sequences in Theorems 1, 2, and 3 are special cases where we know that the conjecture holds. Many new minor special cases where the conjecture holds can be easily deduced from already known results on, e.g., k-extended Skolem sequences. For example, the results in [1] imply that $A = \{1, 2, \ldots, n\} \cup \{2j+1\}$ is a perfect Skolem set for $(n-1)/2 < j \leq n$ if $n \equiv 0, 1 \pmod 4$, and that $A = \{1, 2, \ldots, n\} \cup \{2j\}$ is a perfect Skolem set for $n/2 < j \leq n$ if $n \equiv 2, 3 \pmod 4$. The results in [8] imply that $\{1, 3, 5, \ldots, 2n-1\} \cup \{2, 2j\}$ is a perfect Skolem set for $n \geq 2$ and $2 \leq j \leq n$. We give more evidence for the conjecture in terms of special cases where it can be proved to hold in Section 5.

It is easy to formulate interesting special cases of Conjecture 8. Consider for example sets A of the form $A \subset \{1, 2, \ldots, n\}$, where $|A| = n - 2$, i.e., A is a subset of $\{1, 2, \ldots, n\}$ such that exactly two elements $a_i, a_j \in \{1, 2, \ldots, n\}$ are missing from A.

Conjecture 9 *A subset A of $\{1, 2, \ldots, n\}$ such that exactly two elements $a_i, a_j \in \{1, 2, \ldots, n\}$ are missing from A, is a perfect Skolem set if and only if the following conditions hold.*

(1) $n \equiv 0$ or 1 (mod 4) and $a_i \equiv a_j$ (mod 2), or
(2) $n \equiv 2$ or 3 (mod 4) and $a_i \equiv a_j - 1$ (mod 2); and
(3) $A \notin \{\{3\}, \{2, 4\}, \{2, 4, 5\}, \{3, 4, 5, 6\}\}$.

Note that conditions 1 and 2 correspond to the parity condition in Theorem 7, condition 3 corresponds to the density condition in Theorem 7. The sets in condition 3 are the only ones (of this particular form) that have the right parity but do not satisfy the density condition in Theorem 7.

Another particularly interesting special case of Conjecture 8 emerges when we add the condition

$$\sum_{i=1}^{n} a_i = \left(\sum_{i=n+1}^{2n} p_i\right) - \left(\sum_{i=1}^{n} p_i\right)$$

to the necessary conditions in Theorem 6. We call a multiset A satisfying these conditions *extremal*, and the corresponding sequences *extremal (multi) Skolem-type sequences*. These sets are extremal in the sense that adding 1 (or more) to any of the elements in A would force A to violate the density condition. When $P = \{1, 2, \ldots, 2n\}$ and A is an ordinary set (i.e., not a multiset), then we end up with the following special case of Conjecture 8.

Conjecture 10 *A set $A = \{a_1, a_2, \ldots, a_n\}$ with $a_1 < a_2 < \ldots < a_n$ and $\sum_{i=1}^{n} a_i = n^2$ is a perfect extremal Skolem set if and only if $\sum_{i=m}^{n} a_i \leq n^2 - (m-1)^2$ for each $1 \leq m \leq n$.*

Note that the parity condition in Conjecture 8 is implied by the condition $\sum_{i=1}^{n} a_i = n^2$. Also observe that again the conjecture does not hold when we allow A to be a multiset. A minimal counter example is $\{4, 4, 4, 8, 8, 8\}$. As we will see in Section 4, extremal sets and sequences have interesting connections to problems related to permutation displacements.

3 Enumerative aspects

The problem of deciding how many Langford sequences of a given order there are goes under the name Langford's problem. It is quite well studied, but so far no closed formula is known and the fastest known algorithm runs in exponential time. For an overview of the results; see [9]. Recently Langford's problem has began to be used as a benchmark by the computer science community to evaluate various Constraint Satisfaction techniques [5]. The problem of deciding how many Skolem sequences of a given order there are has not been given the same attention. A table of the number of Skolem sequences of orders 1 to 13 is available in [13]. We extend this table by computing the number of Skolem sequences of orders 16 and 17 (the actual numbers are presented in Table 1).

Table 1
Number of Skolem sequences.

Order	1	4	5	8	9	12
#Sequences	1	6	10	504	2656	455936

Order	13	16	17
#Sequences	3040560	1400156768	12248982496

As for Langford's problem, no closed formula is known and the fastest known algorithm runs in exponential time. We begin our investigation by giving a closed formula for the number of perfect multi Skolem-type sequences of any given order. Note that we say that a sequence is of order n if the set that it is generated from has cardinality n.

Theorem 11 *The number of perfect multi Skolem-type sequences of order n is $(2n-1)!!$.*

PROOF. Think of the sequence as a tape with $2n$ empty cells. Pick 2 cells, say i and j, from the $2n$ available cells. Write the number $|i-j|$ in cells i and j. Now repeat the procedure and choose 2 cells from the remaining $2n-2$ cells. Continue in this way until no more empty cells remains. It is clear that the sequence of numbers on the tape is a perfect multi Skolem-type sequence and that every perfect multi Skolem-type sequence can be constructed in this way. Note that we are not interested in the order in which the n pairs are chosen, we do not care for example if 2 and 6 were chosen before or after 1 and 3. We compensate for this by putting $n!$ in the denominator. Thus, the number of perfect multi Skolem-type sequences of order n is

$$\frac{\binom{2n}{2}\binom{2n-2}{2}\cdots\binom{4}{2}\binom{2}{2}}{n!} = \frac{2n!}{n!2^n} = (2n-1)(2n-3)\ldots(3)(1) = (2n-1)!!$$

□

By using a similar argument as in the proof of the preceding theorem, we can also give a closed formula for the number of perfect extremal multi Skolem-type sequences. An important observation about perfect extremal (multi) Skolem-type sequences is that the first occurrence of each element a_i in A must be placed in the left half of the sequence and the second occurrence must be placed in the right half of the sequence. This is because in any partition (s_i, t_i), $i \in \{1, 2, \ldots, n\}$ of $\{1, 2, \ldots, 2n\}$ into the differences in $\{t_i - s_i\} = A = \{a_1, a_2, \ldots, a_n\}$ satisfying $\sum_{i=1}^{n} a_i = n^2$, we must have $s_i \leq n$ and $t_i > n$ for all $i \in \{1, 2, \ldots, n\}$.

Theorem 12 *The number of perfect extremal multi Skolem-type sequences of order n is $n!$.*

PROOF. In analogy with the proof of the preceding theorem, we pick one cell from the n available cells in the right half of the sequence (tape), say i, and write the number $i - 1$ in cells 1 and i. We repeat the procedure and choose one cell from the remaining $n - 1$ cells in the right half of the sequence, say j, and write the number $j - 2$ in cells 2 and j. We continue in this way until no more empty cells remains. It should be clear that the sequence of numbers on the tape is a perfect extremal multi Skolem-type sequence and that every perfect extremal multi Skolem-type sequence can be constructed in this way. Thus, the number of perfect extremal multi Skolem-type sequences of order n is $n!$. □

Despite our success in finding closed formulas for the number of perfect multi Skolem-type sequences and perfect extremal multi Skolem-type sequences, we have not been as fortunate when it comes to finding a formula for the number of perfect Skolem-type sequences (those constructed from ordinary sets). We have put considerable time and effort in computing the number of perfect Skolem-type sequences of order at most 13. The actual numbers are available in Table 2 and Table 3. In Section 4 we give nice interpretations of these numbers in terms of permutation matrices.

Table 2
Number of perfect Skolem-type sequences.

Order	1	2	3	4	5	6	7	8	9
#Sequences	1	1	5	29	145	957	8397	85169	944221

Order	10	11	12	13
#Sequences	11639417	160699437	2430145085	39776366397

The reader may have noticed that the number of perfect Skolem-type sequences of a given order seems to be odd. This is because perfect Skolem-type sequences are paired via the operation of reversing a sequence, except for the one sequence (of every order) of the form $\ldots 75311357 \ldots$, which is invariant under reversal.

Table 3
Number of perfect extremal Skolem-type sequences.

Order	1	2	3	4	5	6	7	8	9
#Sequences	1	1	3	7	23	83	405	2113	12675

Order	10	11	12	13
#Sequences	82297	596483	4698655	40071743

Moreover, the following conjecture is strongly suggested by the data in Table 2.

Conjecture 13 *The number of perfect Skolem-type sequences of order n is always of the form $4k + 1$.*

Another interesting enumerative question is that of the number of perfect Skolem sets of a given order. If a proof of Conjecture 8 were found it might be possible to find a closed formula for the number of perfect Skolem sets of any given order. Table 4 contains the number of perfect Skolem sets of orders 1 to 20.

Table 4
Number of perfect Skolem sets.

Order	1	2	3	4	5	6	7	8	9
#Sets	1	1	3	11	35	114	407	1486	5414

Order	10	11	12	13	14	15
#Sets	19923	74230	278462	1049318	3972395	15101658

Order	16	17	18	19	20
#Sets	57607431	220391316	845366406	3250192681	12521965697

4 Restricted permutations

In this section we investigate connections between Skolem-type sequences and various problems on permutations. As we will see, extremal Skolem-type sequences will be of particular interest with respect to these problems. Throughout this section S_n denotes the symmetric group of all permutations on the set $\{1, 2, \ldots, n\}$.

We begin by stating some general observations on the connection between Skolem-type sequences and involutions. An involution is a permutation which is its own inverse. A fixed point free involution is an involution where no element is mapped to itself. Another way to express this is that a fixed point free involution is a permutation such that when written in cycle notation it only has cycles of length 2. It is clear that the proof of Theorem 11 gives us a one-to-one correspondence between fixed point free involutions in S_{2n} and perfect multi Skolem-type sequences of order n.

If we restrict the fixed point free involutions and require that all the distances between the transposed elements must be distinct, then we have a one-to-one correspondence between these restricted involutions in S_{2n} and perfect non-multi Skolem-type sequences of order n. For example, the involution (15)(23)(46) satisfies this condition because $|1-5|=4$, $|2-3|=1$ and $|4-6|=2$. Thus, Table 2 gives us the number of these restricted involutions in S_{2n} when $n \leq 13$.

4.1 Permutations and extremal Skolem-type sequences

In this section we will consider an interesting existence question for permutations and its relation to perfect extremal Skolem-type sequences. To the best of our knowledge the following type of fundamental existence question for permutations has not been studied before.

Given a sequence of say 5 elements, is there a permutation of the elements such that one element is moved 4 steps to the right, one element is moved 1 step to the right, one element remains in its original position, one element is moved 2 steps to the left, and one element is moved 3 steps to the left?

We define the problem formally as follows.

Definition 14 *To every permutation π in S_n we associate a displacement pattern α_π defined as follows, let α_π be the multiset of differences $\{\pi(i)-i \mid 1 \leq i \leq n\}$ sorted in increasing order. Given a displacement pattern α we want to determine whether there exists a permutation π having the displacement pattern $\alpha_\pi = \alpha$.*

The question preceding the definition above can now be restated as follows. Does there exist a permutation π in S_5 having the displacement pattern $\alpha_\pi = (-3,-2,0,1,4)$? The answer is yes since for example the permutation $\pi(1) = 5, \pi(2) = 3, \pi(3) = 1, \pi(4) = 4, \pi(5) = 2$ has the displacement pattern $(\pi(5)-5, \pi(3)-3, \pi(4)-4, \pi(2)-2, \pi(1)-1) = (-3,-2,0,1,4)$.

Next we present the connection between the problem of permutation displacements, as defined above, and Skolem-type sequences.

Theorem 15 *Given a displacement pattern $\alpha = (a_1, a_2, \ldots, a_n)$, then there* *exists a permutation π in S_n having the displacement pattern $\alpha_\pi = \alpha$ if and only if the multiset $A = \{a_1+n, a_2+n, \ldots, a_n+n\}$ is a perfect extremal multi Skolem set.*

PROOF. First of all note that the proof of Theorem 12 gives us a one-to-one correspondence between perfect extremal multi Skolem-type sequences of order n and permutations in S_n.

We begin by proving the if part. If the multiset $A = \{a_1+n, a_2+n, \ldots, a_n+n\}$ is a perfect extremal multi Skolem set, then we know that the set $\{1, 2, \ldots, 2n\}$ can be partitioned into the differences in A. Moreover, since A is extremal we know that this partition $a_i+n = t_i - s_i$, $(i = 1, 2, \ldots, n)$ has the property that $s_i \in \{1, 2, \ldots, n\}$ and $t_i \in \{n+1, n+2, \ldots, 2n\}$, $(i = 1, 2, \ldots, n)$. Now consider the permutation π on $\{1, 2, \ldots, n\}$ defined as $\pi(s_i) = t_i - n$, $(i = 1, 2, \ldots, n)$. The corresponding displacement pattern is $\alpha_\pi = (t_1-n-s_1, t_2-n-s_2, \ldots, t_n-n-s_n) = (a_1, a_2, \ldots, a_n)$.

Now to the only if part. If π is a permutation having displacement pattern $\alpha_\pi = (a_1, a_2, \ldots, a_n)$, then $(a_1, a_2, \ldots, a_n) = (\pi(s_1) - s_1, \pi(s_2) - s_2, \ldots, \pi(s_n) - s_n)$ where $\{s_1, s_2, \ldots, s_n\} = \{1, 2, \ldots, n\}$. Hence, letting $t_i = \pi(s_i) + n$, we have that (s_i, t_i) $(i = 1, 2, \ldots, n)$ is a partition of $\{1, 2, \ldots, 2n\}$ into the differences $t_i - s_i$ in $\{a_1+n, a_2+n, \ldots, a_n+n\}$. Thus, $A = \{a_1+n, a_2+n, \ldots, a_n+n\}$ is a perfect extremal multi Skolem set. □

Example 16 *Consider the permutation $\pi \in S_6$ defined by $\pi(1) = 4, \pi(2) = 3, \pi(3) = 5, \pi(4) = 1, \pi(5) = 6, \pi(6) = 2$ having the displacement pattern $\alpha_\pi = (\pi(6)-6, \pi(4)-4, \pi(2)-2, \pi(5)-5, \pi(3)-3, \pi(1)-1) = (-4, -3, 1, 1, 2, 3)$. According to the previous theorem, this implies that $\{6-4, 6-3, 6+1, 6+1, 6+2, 6+3\} = \{2, 3, 7, 7, 8, 9\}$ is a perfect multi Skolem set. Moreover, according to the proof of the previous theorem, one perfect multi Skolem-type sequence witnessing the fact that $\{2, 3, 7, 7, 8, 9\}$ is a perfect multi Skolem set is the sequence 978372327987.*

Note that by the preceding theorem, the existence question for permutation displacements in Definition 14 is merely a special case of the existence question for perfect multi Skolem-type sequences.

The preceding theorem together with the necessary conditions for the existence of perfect multi Skolem-type sequences in Corollary 7 imply the following necessary conditions for the existence of permutations satisfying a given displacement pattern.

Corollary 17 *Given a displacement pattern $\alpha = (a_1, a_2, \ldots, a_n)$, where $a_1 \leq a_2 \leq \ldots \leq a_n$. If there exists a permutation π in S_n having displacement* *pattern $\alpha_\pi = \alpha$, then $\sum_{i=m}^{n} a_i \leq (m-1)(n-m+1)$ for each $1 \leq m \leq n$, with equality holding when $m = 1$.*

Observe that the parity condition in Corollary 7 is redundant here because of the condition $\sum_{i=1}^{n} a_i = 0$. Also note that the bound $(m-1)(n-m+1)$ is derived by noting that $(m-1)(n-m+1) = n^2-(m-1)^2-n(n-m+1)$, where $n^2-(m-1)^2$ is the bound from Corollary 7. These necessary conditions are not sufficient, the displacement pattern $\alpha = (-2,-2,-2,2,2,2)$ is a minimal counterexample.

From Theorem 15 we make the following obvious but important observation.

Corollary 18 *Given a displacement pattern $\alpha = (a_1, a_2, \ldots, a_n)$ where all a_i's are distinct, i.e., $a_1 < a_2 < \ldots < a_n$, then there exists a permutation π in S_n having the displacement pattern $\alpha_\pi = \alpha$ if and only if the set $A = \{a_1+n, a_2+n, \ldots, a_n+n\}$ is a perfect extremal Skolem set.*

Hence, the following conjecture is just a special case of Conjecture 8.

Conjecture 19 *Given a displacement pattern $\alpha = (a_1, a_2, \ldots, a_n)$ where all a_i's are distinct, then the necessary conditions in Theorem 17 are also sufficient for the existence of a permutation π having displacement pattern $\alpha_\pi = \alpha$.*

By the group structure of permutations we get the following result.

Theorem 20 *Given a displacement pattern $\alpha = (a_1, a_2, \ldots, a_n)$, then there exists a permutation π in S_n having the displacement pattern $\alpha_\pi = \alpha$ if and only if there exists a permutation ρ in S_n having the displacement pattern $\alpha_\rho = (-a_n, -a_{n-1}, \ldots, -a_1)$.*

PROOF. Given a permutation π in S_n having displacement pattern $\alpha_\pi = (a_1, a_2, \ldots, a_n)$, then there is a (unique) permutation π^{-1} such that $\pi \circ \pi^{-1} = \iota$, where ι is the identity permutation. Since ι has the displacement pattern $(0, 0, \ldots, 0)$ we get that π^{-1} has displacement pattern $(-a_n, -a_{n-1}, \ldots, -a_1)$. To prove the other direction, just reverse the roles of π and π^{-1}. □

The preceding theorem together with Theorem 15 immediately gives us the following result.

Corollary 21 *$A = \{a_1, a_2, \ldots, a_n\}$ is a perfect extremal multi Skolem set if and only if $A = \{2n-a_n, 2n-a_{n-1}, \ldots, 2n-a_1\}$ is a perfect extremal multi Skolem set.*

Most of the previous research on permutation displacements have focused on enumerative aspects. For example, Lehmer [7] studied the following problems. Determine the number of permutations π in S_n such that

(1) no element is moved more than k positions left or right, that is $|\pi(i)-i| \leq k$ $(i = 1, \ldots, n)$, denoted P_1^k;
(2) the last element is moved to the first position, all other elements are moved right not more than k positions, that is $\pi(n) - n = -n + 1$ and $1 \leq \pi(i) - i \leq k$ $(i = 1, \ldots, n-1)$, denoted P_2^k;
(3) no element is moved more than k positions left or right but each element must move, that is $1 \leq |\pi(i) - i| \leq k$ $(i = 1, \ldots, n)$, denoted P_3^k.

Lehmer gives solutions to $P_1^1, P_1^2, P_1^3, P_2^k, P_3^1$, and P_3^2 in terms of generating functions, e.g., P_2^k is given by the generating function $\frac{1-x-x^2+x^{k+1}}{1-2x+x^k}$. Note that P_3^n is the well known derangement problem which has the solution $n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}$ [16].

It is not hard to see that the correspondence between displacement patterns and perfect extremal multi Skolem sets in Theorem 15 is solution preserving in the sense that the number of permutations having the displacement pattern $\alpha = (a_1, a_2, \ldots, a_n)$ equals the number of perfect extremal multi Skolem-type sequences that can be generated from the multiset $A = \{a_1 + n, a_2 + n, \ldots, a_n + n\}$. Hence, enumerative results on permutation displacements can be transferred to enumerative results on perfect extremal multi Skolem-type sequences, and vice versa. For example, the problem of counting the number of perfect extremal multi Skolem-type sequences of order n having no occurrence of n is just a reformulation of the derangement problem, and thus the solution is $n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}$. Moreover, our count of the number of perfect extremal (non-multi) Skolem-type sequences of order n $(n \leq 13)$ in Table 3 gives us the number of permutations in S_n $(n \leq 13)$ where no two elements are moved the same number of positions in the same direction, or equivalently the number of $n \times n$ permutation matrices $(n \leq 13)$ where each northwest to southeast diagonal contains at most one element. The problem of finding a closed formula (or even a recurrence relation) for the number of these sequences/permutations seems like a very interesting and challenging problem.

We conclude this section with an enumerative result that is an easy consequence of the proof of Theorem 20 and Corollary 21.

Corollary 22 *The number of perfect extremal multi Skolem-type sequences that can be generated from* $A = \{a_1, a_2, \ldots, a_n\}$ *is equal to the number of perfect extremal multi Skolem-type sequences that can be generated from* $A = \{2n - a_n, 2n - a_{n-1}, \ldots, 2n - a_1\}$.

5 Constructions

In this section we show that every perfect multi Skolem set of order n gives rise to a new perfect extremal multi Skolem set of order $2n$. We also show that every k-extended multi Skolem set of order n gives rise to a new perfect extremal multi Skolem set of order $2n+1$. We show how these results can be used to automatically transform many existence results for perfect Skolem sets and k-extended Skolem sets into existence results for perfect extremal Skolem sets. Moreover, we give a complete solution to the existence question for extremal near-Langford sequences.

Theorem 23 *If $A = \{a_1, a_2, \ldots, a_n\}$ is a perfect multi Skolem set, then $B = \{2n - a_n, 2n - a_{n-1}, \ldots, 2n - a_1, 2n + a_1, 2n + a_2, \ldots, 2n + a_n\}$ is a perfect extremal multi Skolem set.*

PROOF. If $A = \{a_1, a_2, \ldots, a_n\}$ is a perfect multi Skolem set, then we know that there exists a partition of $\{1, 2, \ldots, 2n\}$ into the differences in A. As we have seen in Section 4.1 this partition can be seen as a fixed point free involution in S_{2n}. The displacement pattern corresponding to this involution is $(-a_n, -a_{n-1}, \ldots, -a_1, a_1, a_2, \ldots, a_n)$. Now by Theorem 15, we get that $B = \{2n - a_n, 2n - a_{n-1}, \ldots, 2n - a_1, 2n + a_1, 2n + a_2, \ldots, 2n + a_n\}$ is a perfect extremal multi Skolem set. □

Note that if A is a perfect (non-multi) Skolem set, then B is a perfect extremal (non-multi) Skolem set.

Example 24 *We know that $A = \{1, 2, 4\}$ is a perfect Skolem set (as witnessed by the sequence 242114). Hence, by the previous theorem, we know that $B = \{6-4, 6-2, 6-1, 6+1, 6+2, 6+4\} = \{2, 4, 5, 7, 8, 10\}$ is a perfect extremal Skolem set.*

Theorem 25 *If $A = \{a_1, a_2, \ldots, a_n\}$ is a k-extended multi Skolem set, then $B = \{2n+1-a_n, 2n+1-a_{n-1}, \ldots, 2n+1-a_1, 2n+1, 2n+1+a_1, 2n+1+a_2, \ldots, 2n+1+a_n\}$ is a perfect extremal multi Skolem set.*

PROOF. If $A = \{a_1, a_2, \ldots, a_n\}$ is a k-extended multi Skolem set, then we know that there exists a partition of $\{1, 2, \ldots, 2n, 2n+1\} \setminus \{k\}$ into the differences in A. As we have seen in Section 4.1 this partition can be seen as an involution in S_{2n+1} having exactly one fixed point. The displacement pattern corresponding to this involution is $(-a_n, -a_{n-1}, \ldots, -a_1, 0, a_1, a_2, \ldots, a_n)$. Now by Theorem 15, we get that $B = \{2n+1-a_n, 2n+1-a_{n-1}, \ldots, 2n+

$1-a_1, 2n+1, 2n+1+a_1, 2n+1+a_2, \ldots, 2n+1+a_n\}$ is a perfect extremal multi Skolem set. □

Note that if A is a k-extended (non-multi) Skolem set, then B is a perfect extremal (non-multi) Skolem set.

Any perfect multi Skolem set is trivially also a 1-extended multi Skolem set. Hence, the following corollary follows directly from Theorem 25.

Corollary 26 *If $A = \{a_1, a_2, \ldots, a_n\}$ is a perfect multi Skolem set, then $B = \{2n+1-a_n, 2n+1-a_{n-1}, \ldots, 2n+1-a_1, 2n+1, 2n+1+a_1, 2n+1+a_2, \ldots, 2n+1+a_n\}$ is a perfect extremal multi Skolem set.*

The preceding theorems together with known existence results for Skolem-type sequences give us several new existence results for extremal Skolem-type sequences essentially for free. For example, the following result follows from the existence results for near-Skolem sequences and hooked near-Skolem sequences in [12].

Corollary 27 *$A = \{n-1, n, \ldots, 3n-1\} \setminus \{2n-1-m, 2n-1+m\}$ is a perfect extremal Skolem set for all $1 \leq m < n$.*

PROOF. Shalaby proved in [12] that a set of the form $A = \{1, 2, \ldots, m-1, m+1, \ldots, n\}$, where m is even, is a perfect Skolem set for $n \equiv 2, 3 \pmod 4$, and a $(2n-2)$-extended Skolem set for $n \equiv 0, 1 \pmod 4$. Hence, by Theorem 25 and Corollary 26, we get that $A = \{n-1, n, \ldots, 3n-1\} \setminus \{2n-1-m, 2n-1+m\}$, where m is even, is a perfect extremal Skolem set. Shalaby also proved that a set of the form $A = \{1, 2, \ldots, m-1, m+1, \ldots, n\}$, where m is odd, is a perfect Skolem set for $n \equiv 0, 1 \pmod 4$, and a $(2n-2)$-extended Skolem set for $n \equiv 2, 3 \pmod 4$. Again, by Theorem 25 and Corollary 26, we get that $A = \{n-1, n, \ldots, 3n-1\} \setminus \{2n-1-m, 2n-1+m\}$, where m is odd, is a perfect extremal Skolem set. □

In existing terminology, the previous corollary states that there exists a $(2n-1-m,\ 2n-1+m)$-near Langford sequence of length $2n+1$ with defect $n-1$, for all $1 \leq m < n$.

We give some more examples of the same flavour.

Corollary 28 *A is a perfect extremal (multi) Skolem set if*

(1) $A = \{n-2, n-1, \ldots, 3n-2\} \setminus \{2n-2-m, 2n-2, 2n-2+m\}$ and $n \equiv 0, 1 \pmod 4$ and m is odd, or $n \equiv 2, 3 \pmod 4$ and m is even; or

(2) $A = \{(2nm-n)^m, \ldots, (2nm-1)^m, \ldots, (2nm+1)^m, \ldots, (2nm+n)^m\}$ *and* $n \equiv 0, 1 \pmod 4$*, or* $n \equiv 2, 3 \pmod 4$ *and* m *is even.*

PROOF. The proof of (1) follows directly from Theorems 3 and 23, and the proof of (2) follows directly from Theorems 4 and 23. □

Next we show how pairs of perfect extremal multi Skolem sets give rise to new perfect extremal multi Skolem sets.

Theorem 29 *If* $A = \{a_1, a_2, \ldots, a_n\}$ *and* $B = \{b_1, b_2, \ldots, b_m\}$ *are perfect extremal multi Skolem sets, then so are*

(1) $C = A \cup \{b_1 + 2n, b_2 + 2n, \ldots, b_m + 2n\}$*,*
(2) $D = B \cup \{a_1 + 2m, a_2 + 2m, \ldots, a_n + 2m\}$*, and*
(3) $E = \{a_1 + m, a_2 + m, \ldots, a_n + m\} \cup \{b_1 + n, b_2 + n, \ldots, b_m + n\}$*.*

PROOF. If A and B are perfect extremal multi Skolem sets, then they can be used to construct the perfect extremal multi Skolem-type sequences S_A and S_B respectively. If we split S_A in the middle we obtain two sequences S_A^L and S_A^R consisting of the n leftmost and rightmost elements in S_A respectively. It is crucial to realize that since we are working with extremal sequences we know that any two elements that are paired by the partition of $\{1, \ldots, 2n\}$ into two-tuples induced by S_A will have one element in S_A^L and one element in S_A^R. We define S_B^L and S_B^R analogously. Let $S + k$, where S is a sequence and k a natural number, denote the sequence obtained by increasing each element in S by k.

Now, to prove that $C = A \cup \{b_1 + 2n, b_2 + 2n, \ldots, b_m + 2n\}$ is a perfect extremal multi Skolem set, we give a perfect extremal multi Skolem-type sequence S_C which can be generated from $A \cup \{b_1 + 2n, b_2 + 2n, \ldots, b_m + 2n\}$. Let $S_C = (S_B^L + 2n)S_A(S_B^R + 2n)$. Similarly $S_D = (S_A^L + 2m)S_B(S_A^R + 2m)$ is a perfect extremal multi Skolem-type sequence that can be generated from $B \cup \{a_1 + 2m, a_2 + 2m, \ldots, a_n + 2m\}$. Finally, $S_E = (S_A^L + m)(S_B^L + n)(S_A^R + m)(S_B^R + n)$ is a perfect extremal multi Skolem-type sequence which can be created from $\{a_1 + m, a_2 + m, \ldots, a_n + m\} \cup \{b_1 + n, b_2 + n, \ldots, b_m + n\}$. □

We believe that the preceding theorem together with the large number of existence results for perfect extremal Skolem-type sequences, that can automatically be obtained from existence results for (not necessarily extremal) Skolem-type sequences, make them particularly well suited as building blocks for proving existence results for perfect (not necessarily extremal) Skolem-type sequences.

Example 30 *Let*
$A = \{1, 2, \ldots, 4a\}$, $a \geq 1$;
$B = \{b-1, b, \ldots, 3b-1\} \setminus \{2b-1-k, 2b-1+k\}$, $1 \leq k < b$;
$C = \{c+4b-3, \ldots, 3c+4b-3\} \setminus \{2c+4b-3-j, 2c+4b-3+j\}$, $1 \leq j < c$.
Then, $A \cup B \cup C$ is a perfect (multi) Skolem set. By the preceding theorem together with Theorem 27 we know that $B \cup C$ is a perfect (multi) Skolem set as manifested by the perfect (multi) Skolem-type sequence $S_C^L S_B S_C^R$ (using the notation from the proof of the preceding theorem). By Theorem 1, the set A is a perfect Skolem set and by appending the corresponding perfect Skolem sequence S_A to $S_C^L S_B S_C^R$ we get that $A \cup B \cup C$ is a perfect (multi) Skolem set.

The last result of this section is a complete solution to the existence question for extremal near-Langford sequences. Recall that a Langford sequence is perfect Skolem-type sequence that can be generated from a set of the form $A = \{a, a+1, \ldots, b\}$, where a is usually referred to as the *defect* of the sequence/set. Skolem solved the existence question for Langford sequences of defect 1 in [15]; Priday solved it for defect 2 [11]; Bermond et al. solved it for defect 3 and 4 [4]; before Simpson gave a complete solution for all defects [14]. A near-Langford sequence of defect a is perfect Skolem-type sequence that can be generated from a set of the form $A = \{a, a+1, \ldots, b\} \setminus \{m\}$, where $a < m < b$. Shalaby solved the existence question for near-Langford sequences of defect 1 in [12] (see also Theorem 3), the problem is open for all other defects.

Theorem 31 *A set $A = \{a, a+1, \ldots, b\} \setminus \{m\}$, where $a < m < b$, is a perfect extremal Skolem set if and only if $b = 3a$ and $m = (a+b)/2$.*

PROOF. We begin by proving the only if part. Assume (with the aim of reaching a contradiction) that $A = \{a, a+1, \ldots, b\} \setminus \{m\}$ is a perfect extremal Skolem set, but $b \neq 3a$. Two cases emerge

(1) $b > 3a$, i.e., $b = 3a + k$ where $0 < k$, or
(2) $b < 3a$, i.e., $b = 3a - k$ where $0 < k < 2a$.

Recall from the definition of perfect extremal Skolem sets that any perfect extremal Skolem set $A = \{a_1, a_2, \ldots, a_n\}$ must satisfy

$$\sum_{i=1}^{n} a_i = n^2.$$

Hence, we have that $A = \{a, a+1, \ldots, b\} \setminus \{m\}$ must satisfy

$$(\sum_{i=a}^{b} i) - m = (b-a)^2. \quad (3)$$

Now, consider the first case above where $b = 3a + k$, and $0 < k$. Writing the sum in closed form and substituting b by $3a + k$ in (3) above gives us

$$\frac{(4a + k)(2a + k + 1)}{2} - m = (2a + k)^2. \tag{4}$$

By straightforward formula manipulations, we get

$$4a - 2ka + k - k^2 = 2m. \tag{5}$$

The lhs of (5) above is less than or equal to $2a$ for all $k > 0$. This implies that $m \leq a$ for all $k > 0$, which is a contradiction with the fact that $a < m$.

Now, consider the second case above where $b = 3a - k$, and $0 < k < 2a$. By arguments analogous to those in the previous case, we deduce that the equality

$$4a + 2ka - k - k^2 = 2m \tag{6}$$

must hold. We know that m must be less than b and since $b = 3a - k$ we can deduce from (6) that the inequality

$$2(3a - k) > 4a + 2ka - k - k^2 \tag{7}$$

must hold. By simplifying (7), we get $2a > 2ak - k^2 + k$, and by elementary calculus it is easy to see that $2ak - k^2 + k \geq 2a$ for all k in the interval $0 < k < 2a$. Hence, again we have a contradiction and it follows that $b = 3a$.

If $b = 3a$, then we substitute b by $3a$ in (3). A simple calculation verifies that $m = 2a = (a + b)/2$, which concludes the proof of the only if part of the theorem.

To prove the if part of the theorem, we show in Table 5 how to construct perfect extremal Skolem-type sequences from all sets of the form $A = \{a, a + 1, \ldots, b\} \setminus \{m\}$, where $a < m < b$, $b = 3a$, and $m = (a + b)/2$.

Table 5

	a_i	b_i	$\|b_i - a_i\|$	$j \in$
(1)	$2a - 2j$	$3a - j$	$a + j$	$[0, a - 1]$
(2)	$4a + 1 - 2j$	$5a + 1 - j$	$a + j$	$[a + 1, 2a]$

□

6 Final remarks

After having completed this work, the author learned about the recent results in [3]. The problem studied in that paper is easily seen to be equivalent to the existence question for perfect extremal multi Skolem-type sequences. Several interesting connections between this problem and other combinatorial objects are presented in [3]. Moreover, Conjecture 4.2 in [3] is a reformulation of a special case of our Conjecture 8 (in fact, it is equivalent to our Conjecture 10).

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