Frozen Boolean partial co-clones

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Abstract

We introduce and investigate the concept of frozen partial co-clones. Our main motivation for studying frozen partial co-clones is that they have important applications in complexity analysis of constraints. The frozen partial co-clones lie between the co-clones and partial co-clones in the sense that the partial co-clone lattice is a refinement of the frozen partial co-clone lattice, which in turn is a refinement of the co-clone lattice. We concentrate on the Boolean domain and determine large parts of the frozen partial co-clone lattice.

1 Introduction

A clone is a composition closed set of operations containing all projections. An operation f preserves a relation R (or is a *polymorphism* of R) if f applied coordinate-wise to any tuples from R gives a tuple in R. Given a set of relations Γ, $Pol(\Gamma)$ is the set of operations preserving all relations in Γ. $Pol(\Gamma)$ is always a clone and any clone can be defined as $Pol(\Gamma)$ for a set of relations Γ. $Inv(F)$ is the set of relations that are preserved by (invariant under) all the operations in F.

An n-ary relation R has a primitive positive (p.p.) definition (also called an *implementation*) in a set of relations Γ if R is the set of models of an existentially quantified conjunction of atomic formulas over $\Gamma \cup \{=\}$ (also called a Γ-formula), i.e., $R(x_1, \ldots, x_n) \equiv \exists X \bigwedge_i R_i(x_{i1}, \ldots, x_{it})$ where each $R_i \in \Gamma \cup \{=\}$. Sets of relations closed under p.p. definability are called *co-clones* and the least co-clone containing Γ is denoted by $\langle\Gamma\rangle$. There is a Galois connection between clones and sets of relations closed under p.p. definability (i.e., co-clones). In particular, given two sets of relations Γ_1 and Γ_2, then $\langle\Gamma_1\rangle \subseteq \langle\Gamma_2\rangle$ if and only if $Pol(\Gamma_2) \subseteq Pol(\Gamma_1)$, and for any set of relations Γ we have $\langle\Gamma\rangle = Inv(Pol(\Gamma))$. For more information on clones, co-clones, and the aforementioned Galois connection, we refer the reader to the books [13, 15].

The CSP(Γ) problem, where Γ is a set of relations (also called a *constraint language*), is the problem of deciding if a given set of variables subject to a set of constraints (given by atomic formulas over Γ) is satisfiable. The problem of classifying the computational complexity of CSP(Γ) with respect to Γ is an important open problem. The (so far) most successful approach to attack this problem is an algebraic approach which heavily relies on the following result.

Theorem 1 ([10, 11]) *If Γ_1 is finite and $\langle\Gamma_1\rangle \subseteq \langle\Gamma_2\rangle$, then CSP($\Gamma_1$) is polynomial-time reducible to CSP(Γ_2).*

Note that because of the Galois connection between clones and co-clones this result can be reformulated into: If Γ_1 is finite and $Pol(\Gamma_2) \subseteq Pol(\Gamma_1)$, then CSP($\Gamma_1$) is polynomial-time reducible to CSP(Γ_2). To illustrate the power of this result, note that Schaefer's [17] seminal complexity classification (separating the cases in P from the NP-complete cases) for CSP(Γ) over the Boolean domain follows trivially from this result and Post's classification of Boolean clones [14] (i.e., Post's lattice). For more information on the connection between clones, co-clones, and the CSP(Γ) problem, we refer the reader to the survey articles [1, 2, 12].

This result is not so useful for more fine grained complexity analysis of CSP(Γ) (and similar) problems. The reason is that it does not preserve the size of the problem instances (in terms of the number of variables). The crux is that in order for the proof of Theorem 1 to work, the existential quantifiers in p.p. definitions are eliminated by introducing new variables, causing the blow-up in size. As an example, compare 3-SAT (i.e., CSP(Γ_{3SAT}) where Γ_{3SAT} consists of the relations corresponding to clauses on at most three variables) with 1-in-3-SAT (i.e., CSP($\Gamma_{1/3}$) where $\Gamma_{1/3}$ consists of the relation $\{001, 010, 100\}$). By Post's lattice it is easy to verify that $\langle\Gamma_{1/3}\rangle = \langle\Gamma_{3SAT}\rangle$ and hence, CSP(Γ_{3SAT}) is polynomial-time equivalent to CSP($\Gamma_{1/3}$) according to Theorem 1. Despite this, 3-SAT (solvable in time $O(1.473^n)$ [3, 7]) seems to be a much harder problem than 1-in-3-SAT which can be solved in time $O(1.1003^n)$ [4] (where n is the number of variables).

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Hence, it is clear that to get a better understanding of the complexity of CSP(Γ) (and similar) problems, we need reductions/implementations/tools where the blow-up in instance size can be controlled.

With this in mind we consider partial co-clones instead. A partial clone is a composition closed set of *partial* functions containing all (total) projections. A partial operation f preserves a relation R (or is a partial polymorphism of R) if f applied coordinate-wise to any tuples from R gives a tuple in R whenever f is defined on all the arguments. $pPol(\Gamma)$ is the set of (partial) operations preserving all relations in Γ. Partial co-clones can be defined as the sets of relations that are closed under p.p. definitions not using existential quantification, and the least partial co-clone containing Γ is denoted by $\langle\Gamma\rangle_p$. There is also a Galois connection between partial co-clones and partial polymorphisms, in particular we have the following result.

Theorem 2 ([8, 16]) *Let Γ_1 and Γ_2 be sets of relations. Then $\langle\Gamma_1\rangle_p \subseteq \langle\Gamma_2\rangle_p$ if and only if $pPol(\Gamma_2) \subseteq pPol(\Gamma_1)$.*

For formal definitions and more information on partial clones and partial co-clones we refer the reader to [13].

Since partial co-clones do not utilize existential quantification, it can be verified that if we replace co-clones by partial co-clones in Theorem 1 then the reductions are size-preserving (in terms of the number of variables). Hence, if $\langle\Gamma_1\rangle_p \subseteq \langle\Gamma_2\rangle_p$ and Γ_1 is finite then CSP(Γ_1) is solvable in time $O(f(n))$ if CSP(Γ_2) is solvable in time $O(f(n))$ (where n is the number of variables). This result seems more useful than what it is since the structure of partial co-clones is very complicated and only small portions of the lattice are known even for the Boolean domain [9]. We remark in passing that there are other applications of partial clones in complexity analysis of CSP(Γ) problems [18].

What we would like to have is a concept that combines the good features of partial co-clones (e.g., size-preserving reductions for CSP(Γ)) and the good features of co-clones (e.g., simpler structure especially over the Boolean domain). In this paper we propose and investigate such a concept that we call *frozen partial co-clones*. Frozen partial co-clones can be defined as the sets of relations that are closed under p.p. definitions where only the variables that take the same value in every model of the p.p. definition/formula (so called *frozen variables*) can be existentially quantified. The least frozen partial co-clone containing Γ is denoted $\langle\Gamma\rangle_{fr}$. Hence, the frozen partial co-clones lie between the co-clones and partial co-clones in the sense that the partial co-clone lattice is a refinement of the frozen partial co-clone lattice, which in turn is a refinement of the co-clone lattice. Moreover, if we replace co-clones by frozen partial co-clones in Theorem 1 then it can be verified that if $\langle\Gamma_1\rangle_{fr} \subseteq \langle\Gamma_2\rangle_{fr}$ and Γ_1 is finite then CSP(Γ_1) is solvable in time $O(f(n + |D|))$ if CSP(Γ_2) is solvable in time $O(f(n))$ (where D is the domain). The point is that all the variables that are frozen to the same domain element can be replaced by a single variable, and hence, at most $|D|$ extra variables need to be introduced when eliminating the existential quantifiers.

In this paper we focus exclusively on the Boolean domain. As expected, there is a connection between frozen partial co-clones and partial polymorphisms that we sketch (for the Boolean domain) in Section 2. In Sections 3 and 4 we determine large portions of the frozen (Boolean) partial co-clone lattice which indeed is significantly simpler than the partial co-clone lattice.

2 Frozen partial co-clones and partial polymorphisms

In this section we give some definitions and preliminary remarks before sketching the connection between frozen (Boolean) partial co-clones and frozen (Boolean) partial polymorphisms. If φ is a formula, then $Vars(\varphi)$ denotes the set of variables occurring in it, and $\mathcal{M}(\varphi)$ denotes the set of all assignments to $Vars(\varphi)$ which satisfy φ (i.e., the models of φ). The relations $\{0\}$ and $\{1\}$ are denoted by F and T, respectively. Atomic formulas are sometimes written in prefix notation (e.g., $R(x_1, \ldots, x_n)$), or infix notation (e.g., $x_1 \neq x_2$), depending on the context. Given a function f, $dom(f)$ denotes the domain of f (i.e., the set of tuples t_i for which $f(t_i)$ is defined), and f is a *subfunction* of g if $dom(f) \subseteq dom(g)$ and $f(t_i) = g(t_i)$ for all $t_i \in dom(f)$.

Definition 3 (frozen variable) Let φ be a formula and let $x \in Vars(\varphi)$. Then x is said to be *frozen* in φ if $\varphi \models T(x)$ or $\varphi \models F(x)$. In other words, x is frozen in φ if it is assigned the same value by all its models.

Definition 4 (frozen implementation) Let Γ be a set of relations and R an n-ary relation. Then Γ *freezingly implement* R if there is a p.p. definition $R(x_1, \ldots, x_n) \equiv \exists X \varphi$ such that φ is a conjunction of atomic formulas over $\Gamma \cup \{=\}$, $Vars(\varphi) \subseteq X \cup \{x_1, \ldots, x_n\}$, and every variable in X is frozen in φ.

Note that frozen implementations are slightly less general than so-called *faithful implementations* [5, page 34]. But the latter are not suitable for our purposes since they blow up instance sizes.

Definition 5 (frozen partial co-clone) Let Γ be a set of relations. The *frozen partial co-clone* generated by Γ, written $\langle\Gamma\rangle_{fr}$, is the set of all relations that can be freezingly implemented by Γ. Γ is sometimes said to be a *frozen basis* of $\langle\Gamma\rangle_{fr}$.

It is easy to see that frozen implementations compose together, in the sense that if Γ freezingly implements every relation in Γ' and Γ' freezingly implements a relation R, then Γ freezingly implements R. It is also easy to see that frozen partial co-clones ordered by set inclusion form a lattice.

Example 6 The relation $R' = \{00, 10\}$ is in $\langle\{R, T\}\rangle_{fr}$ with $R = \{000, 001, 110\}$ and $T = \{1\}$. This is because $R'(x_1, x_2) \equiv \exists x_3 R(x_2, x_2, x_3) \wedge R(x_2, x_2, x_1) \wedge T(x_3)$ where x_3 is frozen to 1.

As another example, consider the relations $R_1 = \{01, 10\}$ and $R_2 = \{0100, 0110, 1000, 1111\}$. Then $R' = \{010, 011, 100\}$ is in $\langle\{R_1, R_2\}\rangle_{fr}$, as shown by $R'(x_1, x_2, x_3) \equiv \exists x_4 R_1(x_1, x_2) \wedge R_2(x_1, x_2, x_3, x_4)$ (x_4 is frozen to 0 by the conjunction).

Definition 7 (determined) If Γ is a set of relations such that there is a Γ-formula φ in which x is frozen to $d \in \{0, 1\}$, then we say that d is *determined* in Γ.

Proposition 8 *Let Γ be a set of relations. Then $d \in \{0, 1\}$ is determined in Γ if and only if $\{d\} \in \langle\Gamma\rangle_{fr}$.*

Proof: Without loss of generality assume $d = 1$. If 1 is determined in Γ, then there is a Γ-formula φ and a variable $x_T \in Vars(\varphi)$ such that $\exists X\varphi \models T(x_T)$ and thus $\varphi \models T(x_T)$. Let m be a model of φ with a maximum number of variables being assigned 1. Identify all variables in φ that are assigned 1 by m to x_T, resulting in φ'. If $Vars(\varphi') = \{x_T\}$, then $T(x_T) \equiv \varphi'$, and thus $T \in \langle\Gamma\rangle_{fr}$. Otherwise, there is a variable $x_F \in Vars(\varphi') \setminus \{x_T\}$. Identify all variables in $Vars(\varphi') \setminus \{x_T\}$ to x_F, resulting in φ''. Then, $T(x_T) \equiv \exists x_F \varphi''$, and $T \in \langle\Gamma\rangle_{fr}$ since x_F is frozen in φ''. $\Box$

Definition 9 (frozen partial polymorphisms) A k-ary (partial) function $f \in pPol(\Gamma)$ is said to be *frozen* if it is defined on every all-d-tuple (of length k) for which d is determined in Γ and $f(d, d, \ldots, d) = d$. We define $frPol(\Gamma) = \{f \in pPol(\Gamma) \mid f\ is\ frozen\}$, and say that $frPol(\Gamma)$ are the *frozen polymorphisms* of Γ.

Note that if no d is determined in Γ, then $pPol(\Gamma) = frPol(\Gamma)$ and $\langle\Gamma\rangle_p = \langle\Gamma\rangle_{fr}$.

Lemma 10 *Let Γ be a frozen partial co-clone, then any k-ary (partial) function $f \in pPol(\Gamma)$ is a subfunction of a k-ary (partial) function $g \in frPol(\Gamma)$.*

Proof: It is sufficient to prove that if Γ is a frozen partial co-clone such that $d \in \{0, 1\}$ is determined in Γ, then any $f \in pPol(\Gamma)$ is a subfunction of a $g \in pPol(\Gamma)$ such that $\mathbf{d} = (d, d, \ldots, d) \in dom(g)$ and $g(d, d, \ldots, d) = d$. Note that there can be no $f \in pPol(\Gamma)$ such that $f(d, \ldots, d) \neq d$ since this contradicts the fact (observed in Proposition 8) that $\{d\} \in \Gamma$.

Let $f \in pPol(\Gamma)$ be a k-ary function with $dom(f) = \{t_1, \ldots, t_j\}$ such that the k-tuple $\mathbf{d} = (d, \ldots, d)$ is not in $dom(f)$. We define the function f_d with $dom(f_d) = dom(f) \cup \{\mathbf{d}\}$ such that $f_d(\mathbf{d}) = d$ with the goal of showing that $f_d \in pPol(\Gamma)$. Assume to the contrary that there is a relation $R_d \in \Gamma$ such that R_d is not preserved by f_d. This means that there are k (not necessarily distinct) tuples $t_1, \ldots, t_k \in R_d$ such that $(f_d(t_1[1], \ldots, t_k[1]), \ldots, f_d(t_1[m], \ldots, t_k[m])) \notin R_d$ where m is the arity of R_d. By the definition of f_d and the fact that $f \in pPol(\Gamma)$ we know that at least one of the tuples $(t_1[j], \ldots, t_k[j]) = \mathbf{d}$. We can without loss of generality assume an R_d such that $(t_1[1], \ldots, t_k[1]) = \mathbf{d}$ and all other $(t_1[j], \ldots, t_k[j]) \neq \mathbf{d}$.

Since $\{d\} \in \Gamma$ we have that $R(y_1, \ldots, y_{m-1}) \equiv \exists x R_d(x, y_1, \ldots, y_{m-1}) \wedge (x = d)$ is in Γ and thus is preserved by f. As a consequence, we have $(f(t_1[2], \ldots, t_k[2]), \ldots, f(t_1[m], \ldots, t_k[m])) \in R$ and hence, $(d, f(t_1[2], \ldots, t_k[2]), \ldots, f(t_1[m], \ldots, t_k[m])) = (f_d(t_1[1], \ldots, t_k[1]), \ldots, f_d(t_1[m], \ldots, t_k[m])) \in R_d$ and we have a contradiction. Thus, $f_d \in pPol(\Gamma)$. $\Box$

Proposition 11 *Given a set of relations Γ, then $frPol(\Gamma) = frPol(\langle\Gamma\rangle_{fr})$*

Proof: Since $\Gamma \subseteq \langle\Gamma\rangle_{fr}$ we obviously have $frPol(\langle\Gamma\rangle_{fr}) \subseteq frPol(\Gamma)$. For the other direction, assume towards contradiction that $f \in frPol(\Gamma)$ but $f \notin frPol(\langle\Gamma\rangle_{fr})$. Then there is a relation $R \in \langle\Gamma\rangle_{fr}$ such that $f \notin frPol(R)$ and a Γ-formula φ such that $R \equiv \exists X\varphi$ where the variables X are frozen in φ. This is a contradiction since $f \in frPol(\Gamma)$ implies that $f \in frPol(\mathcal{M}(\varphi))$ and hence $f \in frPol(R)$ (since f is frozen). $\Box$

Theorem 12 *Let Γ_1 and Γ_2 be sets of relations, then $\langle\Gamma_1\rangle_{fr} \subseteq \langle\Gamma_2\rangle_{fr}$ if and only if $frPol(\Gamma_2) \subseteq frPol(\Gamma_1)$.*

Proof: If there is a (partial) function f such that $f \in frPol(\Gamma_2)$ and $f \notin frPol(\Gamma_1)$, then assume (with the aim of reaching a contradiction) that $\langle\Gamma_1\rangle_{fr} \subseteq \langle\Gamma_2\rangle_{fr}$. From the fact that $f \in frPol(\Gamma_2)$ and $f \notin frPol(\Gamma_1)$ we get that there is a relation $R \in \Gamma_1$ such that $R \notin \Gamma_2$ and $f \notin pPol(R)$. By assumption $\langle\Gamma_1\rangle_{fr} \subseteq \langle\Gamma_2\rangle_{fr}$, and hence there is a frozen implementation of R using the relations in Γ_2. So, there is a Γ_2-formula φ such that $R \equiv \exists X\varphi$ where the variables in X are frozen in φ. But φ is a Γ_2-formula and $f \in frPol(\Gamma_2)$, so $f \in pPol(\mathcal{M}(\varphi))$. Moreover, as a frozen polymorphism of Γ_2, f is defined on all columns of $\mathcal{M}(\varphi)$ corresponding to variables X (Proposition 8), and finally $f \in pPol(R)$.

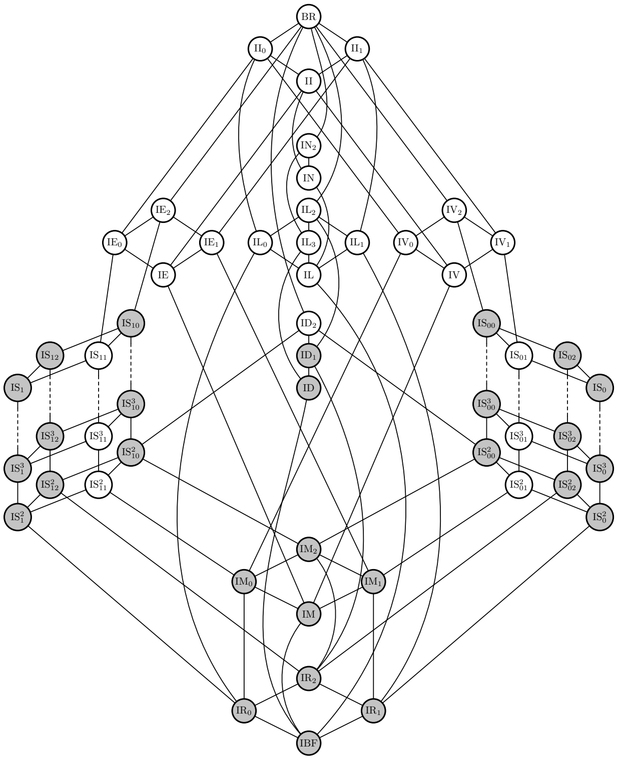

Figure 1. The lattice of Boolean co-clones.

Now we continue with the other direction. If $\langle\Gamma_1\rangle_{fr} \not\subseteq \langle\Gamma_2\rangle_{fr}$, then trivially $\langle\langle\Gamma_1\rangle_{fr}\rangle_p \not\subseteq \langle\langle\Gamma_2\rangle_{fr}\rangle_p$. Hence, by Theorem 2 there exists a (partial) function f' such that $f' \in pPol(\langle\Gamma_2\rangle_{fr})$ and $f' \notin pPol(\langle\Gamma_1\rangle_{fr})$. By Lemma 10, we have that f' is a subfunction of a (partial) function $f \in frPol(\langle\Gamma_2\rangle_{fr})$. It is clear that $f \notin frPol(\langle\Gamma_1\rangle_{fr})$ since not even the subfunction f' is in $pPol(\langle\Gamma_1\rangle_{fr})$. Hence, by Proposition 11, we get $f \in frPol(\Gamma_2)$ and $f \notin frPol(\Gamma_1)$. □

Corollary 13 *Let* Γ *be a set of relations. Then* $Inv(frPol(\Gamma)) = \langle\Gamma\rangle_{fr}$.

3 Co-clones covered by a single frozen partial co-clone

In this section we study Boolean co-clones C such that for any set of relations Γ with $\langle\Gamma\rangle = C$ we have $C = \langle\Gamma\rangle_{fr}$. We say that such a co-clone is *covered by a single frozen partial co-clone*. We show that a large number of co-clones are covered by a single frozen partial co-clone, and hence a large part of the lattice of frozen partial co-clones is identical to the corresponding part of the lattice of co-clones. The lattice of Boolean co-clones is visualised in Figure 1, where the co-clones colored grey are covered by a single frozen partial co-clone. For explanations of the notation[1] used in the lattice, we refer the reader to [1, 2].

We define the most relevant co-clones here. A majority operation is a ternary operation maj satisfying $maj(x,x,y) = maj(x,y,x) = maj(y,x,x) = x$ for all $x,y \in \{0,1\}$. Similarly a minority operation is a ternary operation $minor$ satisfying $minor(x,x,y) = minor(x,y,x) = minor(y,x,x) = y$ for all $x,y \in \{0,1\}$. The binary operations max and min return the maximum and minimum of their arguments, respectively. The co-clones ID_2, ID_1, and IM_2 are defined as follows: $ID_2 = Inv(\{maj\})$, $ID_1 = Inv(\{maj, minor\})$, and $IM_2 = Inv(\{max, min\})$.

In [9] it is proved that there are 25 partial co-clones[2] pC such that $pC \subseteq IM_2 = Inv(\{max, min\})$, and there are 33 partial co-clones pC such that $pC \subseteq ID_1 = Inv(\{maj, minor\})$. Since all co-clones C such that $C \subseteq IM_2$ or $C \subseteq ID_1$ are covered by a single frozen partial co-clone, there are only 8 frozen partial co-clones $fC \subseteq IM_2$ and 6 frozen partial co-clones $fC \subseteq ID_1$. This suggests that the lattice of frozen partial co-clones is significantly less complex than the partial co-clone lattice. Nevertheless, as expected the frozen partial co-clone lattice is more complex than the ordinary co-clone lattice. In particular we show in Section 4 that the co-clone ID_2 splits into 13 frozen partial co-clones. Moreover, it seems that none of the white co-clones in Figure 1 is covered by a single frozen partial co-clone.

The covering proofs make heavy use of the results in [6] which for every Boolean co-clone C gives a set of relations Γ such that $\langle\Gamma\rangle_p = C$. In particular, it is shown in [6] that $ID_1 = \langle\{\neq, T, F\}\rangle_p$. The covering proofs are numerous so due to space constraints we can only present a single illustrative case.

Proposition 14 $ID_1 = Inv(\{maj, minor\}) = \langle\{\neq, T, F\}\rangle_p$ *is covered by a single frozen partial co-clone.*

Proof: Let $\Gamma \subseteq ID_1$ with $\Gamma \not\subseteq ID$ and $\Gamma \not\subseteq IR_2$ (i.e., $\langle\Gamma\rangle = ID_1$), where $ID = \langle\neq\rangle$ and $IR_2 = \langle\{F, T\}\rangle$. First note that since $\langle\Gamma\rangle = ID_1 = \langle\{\neq, T, F\}\rangle$, we trivially have that F and T are determined in Γ. Thus, according to Proposition 8 we have $\{T, F\} \subseteq \langle\Gamma\rangle_{fr}$.

Now, since $\Gamma \not\subseteq IR_2$ there is a relation $R \in \Gamma$ such that $R \notin IR_2$. Then there is $\varphi \equiv R$ of the form:

$$\bigwedge_{i\in I}(x_i \neq y_i) \wedge \bigwedge_{j\in J} F(x_j) \wedge \bigwedge_{k\in K} T(x_k)$$

Assume no j and no k is in I, which can be ensured by propagating unary constraints. From $R \notin IR_2$ we know

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[1] The notation used is different from Post's notation, but is now standard in the Boolean CSP area.

[2] The results in [9] are presented in terms of partial clones.

that R is nonempty and $I \neq \emptyset$. Let $m \in R$, that is, a model of φ, and let $P = \{x_i \mid i \in I, m(x_i) = 1\} \cup \{y_i \mid i \in I, m(y_i) = 1\}$ and $N = \{x_i \mid i \in I, m(x_i) = 0\} \cup \{y_i \mid i \in I, m(y_i) = 0\}$. Because $I \neq \emptyset$ we have at least one $\{\neq\}$-constraint and so, $P, N \neq \emptyset$. Moreover, obviously every $\{\neq\}$-constraint in φ is between a variable in P and one in N. Now identify all the variables in P to a single variable p and all those in N to n. Then clearly the resulting formula is logically equivalent to $(p \neq n) \wedge \bigwedge_{j \in J} F(x_j) \wedge \bigwedge_{k \in K} T(x_k)$, and we get a frozen implementation of $\neq$ by existentially quantifying over every x_j and x_k ($j \in J, k \in K$).

Finally, $\{\neq, T, F\} \subseteq \langle\Gamma\rangle_{fr}$ and so, $\langle\Gamma\rangle_{fr} = ID_1$. $\square$

Theorem 15 *Each co-clone colored grey in Figure 1 is covered by a single frozen partial co-clone.*

4 Structure of ID$_2$

We begin by introducing the basic relations and the 13 frozen partial co-clones in $ID_2 = Inv(\{maj\})$ (i.e., the frozen partial co-clones $\langle\Gamma\rangle_{fr}$ such that $\langle\Gamma\rangle_{fr} \subseteq ID_2$ and $\langle\Gamma\rangle = ID_2$). We then prove that these 13 frozen partial co-clones cover ID_2 (i.e., $\langle\Gamma\rangle_{fr}$ equals one of these 13 frozen partial co-clones for any Γ such that $\langle\Gamma\rangle = ID_2$). Finally, we prove that these 13 frozen partial co-clones are all distinct. We remark that the lattice of partial co-clones in ID_2 has not yet been classified [9]. Hence, the results in this section can also be seen as a step towards such a classification.

Definition 16 (relations in ID_2) R_2^p is the relation defined by $(x_1 \vee x_2)$ (2 stands for *binary* and p for *positive*). Similarly, R_2^n is the relation defined by $(\neg x_1 \vee \neg x_2)$, R_2^i is the relation defined by $(\neg x_1 \vee x_2)$ (i stands for *implicative*), and $R_2^{\neq}$ is the relation defined by $x_1 \neq x_2$. R_3^p is the relation defined by $(x_1 \vee x_2) \wedge (x_1 \neq x_3)$ and R_3^n that defined by $(\neg x_1 \vee \neg x_2) \wedge (x_1 \neq x_3)$. Finally, R_4^p is the relation defined by $(x_1 \vee x_2) \wedge (x_1 \neq x_3) \wedge (x_2 \neq x_4)$.

Definition 17 [frozen partial co-clones in ID_2] We define the following frozen partial co-clones:

- $\Gamma_4^p = \langle R_4^p \rangle_{fr}$,
- $\Gamma_3^p = \langle R_3^p \rangle_{fr}$, $\Gamma_3^n = \langle R_3^n \rangle_{fr}$,
- $\Gamma_3^{np} = \langle R_3^n, R_3^p \rangle_{fr}$,
- $\Gamma_2^{n\neq} = \langle R_2^n, R_2^{\neq} \rangle_{fr}$, $\Gamma_2^{p\neq} = \langle R_2^p, R_2^{\neq} \rangle_{fr}$,
- $\Gamma_2^{i\neq} = \langle R_2^i, R_2^{\neq} \rangle_{fr}$, $\Gamma_2^{np} = \langle R_2^n, R_2^p \rangle_{fr}$,
- $\Gamma_{23}^{np} = \langle R_2^n, R_3^p \rangle_{fr}$, $\Gamma_{23}^{pn} = \langle R_2^p, R_3^n \rangle_{fr}$,
- $\Gamma_2^{n\neq i} = \langle R_2^n, R_2^{\neq}, R_2^i \rangle_{fr}$, $\Gamma_2^{p\neq i} = \langle R_2^p, R_2^{\neq}, R_2^i \rangle_{fr}$,

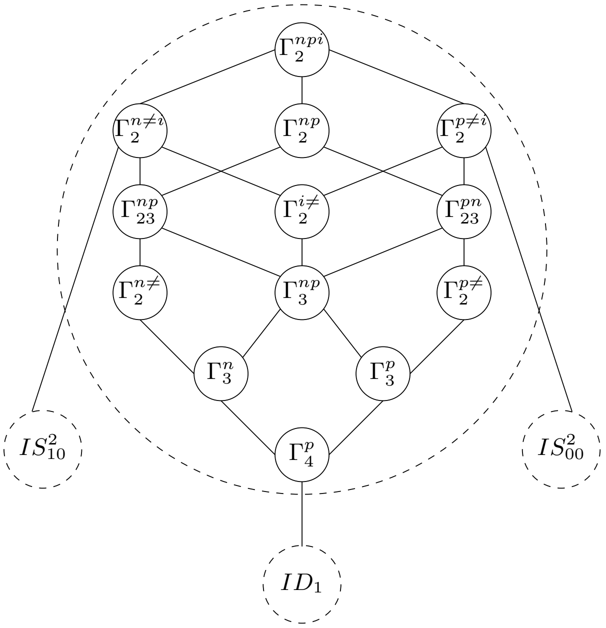

Figure 2. Frozen partial co-clones in ID_2.

- $\Gamma_2^{npi} = \langle R_2^n, R_2^p, R_2^i \rangle_{fr}$. [3]

The inclusion structure among these frozen partial co-clones is given in Figure 2. Most of the inclusions are obvious. The main difficulty is to show that they cover all of ID_2. Due to space constraints and the fact that all the covering proofs are quite similar, we only present one of them. Note that the proof is similar in spirit to the constructions in [18].

Proposition 18 (Γ_4^p) *Let R be a relation in $\Gamma_4^p \setminus ID_1$. Then R freezingly implements R_4^p.*

Proof: Given a relation $R \equiv \mathcal{M}(\varphi)$, then $R_{|\{x_i,\ldots,x_j\}}$ denotes the projection of R onto the coordinates corresponding to the variables $\{x_i, \ldots, x_j\}$ in φ.

From $R \in \Gamma_4^p$ it follows that R has a p.p. definition of the form $\exists X, \bigwedge_{i \in I} R_4^p(x_{i1}, \ldots, x_{i4})$ where the variables in X are frozen. Write R_i for $R_{|\{x_{i1},\ldots,x_{i4}\}}$ and $R_4^p(X_i)$ for $R_4^p(x_{i1}, \ldots, x_{i4})$. We claim that there is an $i_0 \in I$ such that $R_{i_0} = R_4^p(X_{i_0})$ and all $x_{i_0 j}$'s are pairwise different.

Assume to the contrary that for all i, $R_i \neq R_4^p(X_i)$. By construction it follows $R_i \subset R_4^p(X_i)$ (since at least the $\{R_4^p\}$-constraint acts on $x_{i1}, \ldots, x_{i4}$). But a case study on the tuples in $R_4^p(X_i) \setminus R_i$ shows that this entails $R_i \in ID_1$. Since from the definition of R it follows $R \equiv \exists X, \bigwedge_{i \in I} R_i$, we get $R \in ID_1$, a contradiction.

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[3] The mnemonics are: subscripts represent the arities of the relations in the basis, and superscripts represent the nature of these relations, in the same order (p stands for *positive*, etc.).

Now consider the relation obtained from R by applying the following transformations maximally while preserving $R_{i_0} = R_4^p(X_{i_0}) = \{0110, 1001, 1100\}$:

1. identify x' to x for some $x \in \{x_{i_01}, \dots, x_{i_04}\}$, $x' \notin \{x_{i_01}, \dots, x_{i_04}\}$; e.g., if $R_{|\{x_{i_01},\dots,x_{i_04},x'\}} = \{01100, 01101, 10011, 11001\}$, identify x' to x_{i_01};
2. freeze $x' \notin \{x_{i_01}, \dots, x_{i_04}\}$ to 0 or 1 (using F or T with Proposition 8) and existentially quantify over it, e.g., if $R_{|\{x_{i_01},\dots,x_{i_04},x'\}} = \{01100, 01101, 10010, 11000\}$, freeze x' to 0 and existentially quantify over it.

When none of these operations can be applied any more, it can be easily verified that all remaining $x' \notin \{x_{i_01}, \dots, x_{i_04}\}$ are such that $R_{|\{x_{i_01},\dots,x_{i_04},x'\}} = \{01101, 10011, 11000\}$ or $R_{|\{x_{i_01},\dots,x_{i_04},x'\}} = \{01100, 10010, 11001\}$. But this is a contradiction, since in this case $R_{|\{x_{i_01},\dots,x_{i_04},x'\}}$ is not closed under ternary majority, and so $R \notin ID_2$. Thus the transformations end with $R = R_4^p(X_{i_0})$ (no other x_j can be left).

To conclude, from R we freezingly implemented R_4^p, and we are done. $\square$

Theorem 19 *Given any set of relations* Γ *such that* $\langle\Gamma\rangle = ID_2$*, then* $\langle\Gamma\rangle_{fr}$ *equals one of the* 13 *frozen partial co-clones in Definition 17.*

It is easy to prove that the 13 frozen partial co-clones in Definition 17 are all distinct. Given two sets of relations Γ_1 and Γ_2 such that $\langle\Gamma_1\rangle = \langle\Gamma_2\rangle = ID_2$, then to prove that $\langle\Gamma_1\rangle_{fr} \neq \langle\Gamma_2\rangle_{fr}$ it is sufficient (according to Theorem 12) to show that $frPol(\Gamma_2) \neq frPol(\Gamma_1)$.

The operations that we use to separate the different frozen partial co-clones are all ternary minority operations that are undefined on certain tuples. We denote these minority operations by $mU(t_1, \dots, t_n)$ where $t_1, \dots, t_n$ are the tuples on which the minority operation is undefined. For example, $\Gamma_4^p \neq \Gamma_3^p$ since $mU(010, 001) \in frPol(R_4^p) = frPol(\Gamma_4^p)$ but $mU(010, 001) \notin frPol(R_3^p)$, and $\Gamma_3^p \neq \Gamma_2^{p\neq}$ since $mU(100, 010, 001) \in frPol(R_3^p) = frPol(\Gamma_3^p)$ but $mU(100, 010, 001) \notin frPol(R_2^p)$.

Proposition 20 *The* 13 *frozen partial co-clones in Definition 17 are all distinct.*

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